Basic Integration
10.2 – U Substitution Indefinite Integrals

Substitution Method

Using differential notation, we can state three very useful integration formulas.

(A) $\int u^n du = \frac{1}{n+1} u^{n+1} + C \qquad n \neq -1$

(B) $\int e^u du = e^u + C$

(C) $\int \frac{1}{u} du = \ln|u| + C$

Why Integration By Substitution?

Many functions cannot be integrated directly. Some functions can be integrated if a "u-substitution" is first done.

Differential

For a differentiable function *f(x)*, the differential *df* is

$$df = f'(x)\, dx$$

Explanation for Differential

One of the notations for the derivative of a function *f(x)* is $\frac{df}{dx}$. Although written as a fraction, $\frac{df}{dx}$ was not defined as the quotient of two quantities *df* and *dx*, but as a single object, the *derivative*. We will now define *df* and *dx* separately (they are called differentials) so that their quotient *df ÷ dx* is equal to the derivative $\frac{df}{dx}$.

$$f = f$$

$$\frac{df}{dx} = f'$$

$$df = f'\,dx$$

Note that df does NOT mean *d* times *f*. The dx is just the notation that appears at the end of integrals, arising from the Δx in the Riemann sum. The reason for finding the differentials will be made clear shortly.

Baby Step 1 – Solve for the differential *du*

#1) $u = 3x^2$

$$\frac{du}{dx} = 6x$$

$$du = 6x\, dx$$

> #1) Differentiate each side with respect to x.
>
> #2) Solve equation for du.

#2) $u = \ln x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{1}{x}\, dx$$

#3) $u = \frac{1}{2} e^{x^2}$

$$\frac{du}{dx} = x e^{x^2}$$

$$du = x e^{x^2}\, dx$$

Baby Step 2 – Solve for the differential *dx*

#1) $u = x^3 + 1$

$$\frac{du}{dx} = 3x^2$$

$$du = 3x^2\, dx$$

$$\frac{du}{3x^2} = dx$$

> #1) Differentiate each side with respect to x.
>
> #2) Solve equation for dx.

#2) $u = e^{2t} + 1$

$$\frac{du}{dt} = 2e^{2t}$$

$$du = 2e^{2t}\, dt$$

$$\frac{du}{2e^{2t}} = dt$$

#3) $u = e^{-5t}$

$$\frac{du}{dt} = -5e^{-5t}$$

$$du = -5e^{-5t}\, dt$$

$$\frac{du}{-5e^{-5t}} = dt$$

Substitution Method (Repeated)

(A) $\int u^n du = \frac{1}{n+1} u^{n+1} + C$ $n \neq -1$

(B) $\int e^u du = e^u + C$

(C) $\int \frac{1}{u} du = \ln|u| + C$

Baby Step 3 – For each of the following integrals, choose the most appropriate formula: (A), (B), or (C).

#1) $\int e^{5x^2 - 1} x\, dx$

B

#2) $\int \frac{x\, dx}{x^2 + 1}$

C

#3) $\int (x^4 - 12)^4 x^3 dx$

A

#4) $\int (x^4 - 12)^{-1} x^3 dx$

C

Integrating by Substitution

#1) $\int (x^2 + 2)^3 2x\, dx$

$= \int (u)^3 \, 2x \left(\frac{du}{2x} \right)$

$= \int u^3 du$

$= \frac{1}{4} u^4 + C$

$= \frac{1}{4}(x^2 + 2)^4 + C$

$u = x^2 + 2$

$\frac{du}{dx} = 2x$

$du = 2x\, dx$

$\frac{du}{2x} = dx$

#1) Set u equal to an expression. (Be smart!)

#2) Differentiate both sides of that equation and solve for dx.

#3) Substitute in u and substitute for dx.

#4) Simplify

#5) Integrate the function.

#6) Finally, replace the u with the expression from step #1.

#2) $\int e^{x^2 - 4} 2x\, dx$

$= \int e^u \, 2x \left(\frac{du}{2x} \right)$

$= \int e^u du$

$= e^u + C$

$= e^{x^2 - 4} + C$

$u = x^2 - 4$

$\frac{du}{dx} = 2x$

$du = 2x\, dx$

$\frac{du}{2x} = dx$

#3) $\int \frac{3x^2}{x^3 - 7} dx$

$= \int \frac{3x^2}{u} \left(\frac{du}{3x^2} \right)$

$= \int \frac{1}{u} du$

$= \ln|u| + C$

$= \ln|x^3 - 7| + C$

$u = x^3 - 7$

$\frac{du}{dx} = 3x^2$

$du = 3x^2 dx$

$\frac{du}{3x^2} = dx$

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Integrate by substitution with extra constants.

#4) $\int (x^2 + 4)^3 x \, dx = \int u^3 x \left(\frac{du}{2x} \right)$

$= \frac{1}{2} \int u^3 \, du$

$= \frac{1}{2} \left(\frac{1}{4} u^4 \right) + C$

$= \frac{1}{8} (x^2 + 4) + C$

$u = x^2 + 4$

$\frac{du}{dx} = 2x$

$du = 2x \, dx$

$\frac{du}{2x} = dx$

#1) $\int \sqrt{x^3 - 3x}(x^2 - 1) dx = \int \sqrt{u} \, (x^2 - 1) \left(\frac{du}{3x^2 - 3} \right)$

$= \int \sqrt{u} \, (x^2 - 1) \frac{du}{3(x^2 - 1)}$

$= \frac{1}{3} \int u^{\frac{1}{2}} \, du$

$= \frac{1}{3} \left(\frac{2}{3} \right) u^{3/2} + C$

$= \frac{2}{9} \left(\sqrt{x^3 - 3x} \right)^3 + C$

$u = x^3 - 3x$

$\frac{du}{dx} = 3x^2 - 3$

$du = (3x^2 - 3) dx$

$\frac{du}{3x^2 - 3} = dx$

#2) $\int e^{\sqrt{x}} x^{-1/2} dx = \int e^u \frac{1}{\sqrt{x}} \left(2\sqrt{x} \, du \right)$

$= 2 \int e^u \, du$

$= 2 e^u + C$

$= 2 e^{\sqrt{x}} + C$

$u = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$

$du = \frac{1}{2\sqrt{x}} dx$

$2\sqrt{x} \, du = dx$

Substitution Does NOT Work For All Problems
Show that substitution cannot be used to integrate

$\int e^{x^4} dx = \int e^u \left(\frac{du}{4x^3} \right)$

$u = x^4$

$\frac{du}{dx} = 4x^3$

$du = 4x^3 dx$

$\frac{du}{4x^3} = dx$

Baby Step 4 – Decide Which Integral Can Be Found By Substitution

#1) $\int (x^3 + 1)^3 x^3 dx = \int u^3 x^3 \left(\frac{du}{3x^2} \right)$ NO

$u = x^3 + 1$

$\frac{du}{dx} = 3x^2$

$du = 3x^2 dx$

$\frac{du}{3x^2} = dx$

#2) $\int e^{x^2} dx = \int e^u \left(\frac{du}{2x} \right)$

NO

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x dx$

$\frac{du}{2x} = dx$