10.3 – U Substitution Definite Integrals

A: Integrate each definite integral with a u-substitution.

#1) $\int_0^2 (x^2-1)^5 2x\,dx$

$u = x^2 - 1$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$= \int_{-1}^{3} u^5 \cancel{2x}\left(\frac{du}{\cancel{2x}}\right)$

$= \int_{-1}^{3} u^5\,du$

$= \frac{1}{6}u^6 \Big|_{-1}^{3}$

$= \left[\frac{1}{6}(3)^6\right] - \left[\frac{1}{6}(-1)^6\right]$

$= \left[\frac{1}{6}(729)\right] - \left[\frac{1}{6}(1)\right]$

$= \frac{729}{6} - \frac{1}{6}$

$= \frac{728}{6}$

$= \frac{364}{3}$

#2) $\int_0^1 e^{x^2} 2x\,dx$

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$= \int_0^1 e^u \cancel{2x}\left(\frac{du}{\cancel{2x}}\right)$

$= \int_0^1 e^u\,du$

$= e^u \Big|_0^1$

$= \left[e^1\right] - \left[e^0\right]$

$= e - 1$

#3) $\int_{-3}^{3} \frac{2x}{x^2-1}\,dx$

$u = x^2 - 1$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$= \int_8^8 \frac{\cancel{2x}}{u}\left(\frac{du}{\cancel{2x}}\right)$

$= \int_8^8 \frac{1}{u}\,du$

$= \ln|u| \Big|_8^8$

$=$

#4) $\int_{-1}^{1} (x^2-1)^5 x\,dx$

$u = x^2 - 1$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$= \int_0^0 u^5 \cancel{x}\left(\frac{du}{2\cancel{x}}\right)$

$= \frac{1}{2}\int_0^0 u^5\,du$

$= \frac{1}{12}u^6 \Big|_0^0$

$=$

#5) $\int_1^2 e^{x^3} 3x^2 \, dx$

$u = x^3$
$\frac{du}{dx} = 3x^2$
$du = 3x^2 dx$
$\frac{du}{3x^2} = dx$

$= \int_1^8 e^u \cancel{3x^2}\left(\frac{du}{\cancel{3x^2}}\right)$

$= \int_1^8 e^u du$

$= e^u \Big|_1^8$

$= e^8 - e^1$

$= e^8 - e$

#6) $\int_{-2}^{2} \frac{x^2}{x^3+2} dx = \int_{-6}^{10} \frac{\cancel{x^2}}{u}\left(\frac{du}{3\cancel{x^2}}\right)$

$u = x^3 + 2$
$\frac{du}{dx} = 3x^2$
$du = 3x^2 dx$
$\frac{du}{3x^2} = dx$

$= \frac{1}{3}\int_{-6}^{10} \frac{1}{u} du$

$= \frac{1}{3} \ln|u| \Big|_{-6}^{10}$

$= \frac{1}{3} \ln|u| \Big|_{-6}^{10}$

$= \left[\frac{1}{3}\ln|10|\right] - \left[\frac{1}{3}\ln|-6|\right]$

$= \frac{1}{3}\left[\ln 10 - \ln 6\right]$

$= \frac{1}{3}\ln\left(\frac{5}{3}\right)$

Bathroom Tissue

#7) Adding to his line of products for *The Slightly Used Company*, George starts selling bathroom tissue. *Slightly Used's* marginal cost function is $MC(x) = \frac{1}{4x+2}$ and its fixed costs are \$4. Find the cost function.

$(0, 4)$

$C(x) = \int MC(x) dx$

$= \int \frac{1}{4x+2} dx$

$= \int \frac{1}{u}\left(\frac{du}{4}\right)$

$= \frac{1}{4}\int \frac{1}{u} du$

$= \frac{1}{4}\ln|u| + C$

$C(x) = \frac{1}{4}\ln|4x+2| + C$

$u = 4x + 2$
$\frac{du}{dx} = 4$
$du = 4dx$
$\frac{du}{4} = dx$

$4 = \frac{1}{4}\ln|4(0)+2| + C$
$4 = \frac{1}{4}\ln|2| + C$
$16 = \ln 2 + C$
$16 - \ln 2 = C$

$C(x) = \frac{1}{4}\ln|4x+2| + 16 - \ln 2$

Pluckable Hairs

#8) The number of pluckable hairs on George's ears is expected to be $P(x) = x(x^2+4)^{-1/2}$ hairs after x months. Find the average number of pluckable hairs between month $x = 0$ and month $x = 8$.

Average Pluckable Hairs $= \frac{1}{8-0}\int_0^8 x(x^2+4)^{-\frac{1}{2}} dx$

$u = x^2 + 4$
$\frac{du}{dx} = 2x$
$du = 2x dx$
$\frac{du}{2x} = dx$

$= \frac{1}{8}\int_4^{68} \cancel{x}\, u^{-\frac{1}{2}}\left(\frac{du}{2\cancel{x}}\right)$

$= \frac{1}{8}\cdot\frac{1}{2}\int_4^{68} u^{-\frac{1}{2}} du$

$= \frac{1}{8} u^{\frac{1}{2}} \Big|_4^{68}$

$= \left[\frac{1}{8}\sqrt{68}\right] - \left[\frac{1}{8}\sqrt{4}\right]$

$= \frac{1}{8}\sqrt{68} - \frac{1}{4}$

Average Pluckable Hairs $\approx .78$ per month

The average number of pluckable hairs from month 0 to 8 is .78 per month.

Alliteration – The Prequel

#9) George sells sails for snail sized sailboats. His sales of sails for snail sized sailboats during week x are given by $S(x) = \frac{1}{x+4}$ in hundreds. Find the average sales of sails for snail sized sailboats from week $x = 1$ to week $x = 4$. (Don't forget your answer is in hundreds, noob.)

$$AS = \frac{1}{4-1}\int_1^4 \frac{1}{x+4}\,dx$$

$$\boxed{\begin{aligned} u &= x+4 \\ \frac{du}{dx} &= 1 \\ du &= dx \end{aligned}}$$

$$= \frac{1}{4}\int_5^8 \frac{1}{u}\,du$$

$$= \frac{1}{4}\ln|u|\Big|_5^8$$

$$= \left[\frac{1}{4}\ln|8|\right] - \left[\frac{1}{4}\ln|5|\right]$$

$$= \frac{1}{4}\left[\ln 8 - \ln 5\right]$$

$= \frac{1}{4}\ln\frac{8}{5}$ hundred

$= .12$ hundred

$= 12$

The average sales of sails was 12 from week 1 to 4.

Alliteration

#10) An experimental therapy lowers a patient's patience for patterns at the rate of $t\sqrt{36 - t^2}$ units per day, where t is the number of days since the therapy was administered (for the first six days). Find the total change in a patient's patience for patterns during the first 3 days.

$$PPP = \int_0^3 t\sqrt{36-t^2}\,dt$$

$$\boxed{\begin{aligned} u &= 36-t^2 \\ \frac{du}{dt} &= -2t \\ du &= -2t\,dt \\ \frac{du}{-2t} &= dt \end{aligned}}$$

$$= \int_{36}^{27} \cancel{t}\sqrt{u}\left(\frac{du}{-2\cancel{t}}\right)$$

$$= -\frac{1}{2}\int_{36}^{27} u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\int_{27}^{36} u^{1/2}\,du$$

$$= \frac{1}{2}\left(\frac{2}{3}\right)u^{3/2}\Big|_{27}^{36}$$

$$= \left[\frac{1}{3}\left(\sqrt{36}\right)^3\right] - \left[\frac{1}{3}\left(\sqrt{27}\right)^3\right]$$

$$= \left[\frac{1}{3}(6)^3\right] - \left[\frac{1}{3}\sqrt{27}^3\right]$$

$$= \frac{216}{3} - \frac{\left(\sqrt{27}\right)^3}{3}$$

$$= \frac{216 - \left(\sqrt{27}\right)^3}{3}$$

≈ 25.235 (CALC)

A patient's patience for patterns lowers by about 25 units during the first 3 days.

Condiments

#11) George has developed a new business model for making money in the restaurant business – give away the food for free, but charge for the condiments. He is selling condiments at the rate of $100e^{-x}$ per week after x weeks. How many condiments will be sold during the first 8 weeks?

$$C = \int_0^8 100e^{-x}\,dx$$

$$\boxed{\begin{array}{l} u = -x \\ \frac{du}{dx} = -1 \\ du = -dx \\ -du = dx \end{array}}$$

$$= \int_0^{-8} 100e^u(-du)$$

$$= 100\int_{-8}^{0} e^u\,du$$

$$= 100e^u\Big|_{-8}^{0}$$

$$= [100e^0] - [100e^{-8}]$$

$$= 100 - \frac{100}{e^8}$$

$C \approx 100$ condiments sold CALC

George will sell 100 condiments during the first week.

Discharging Pits

#12) George's armpits are discharging pollution into the air at the rate of $r(t)$ liters per year given by $r(t) = \frac{1}{t+1}$ where t is the number of years since George washed. Find the total amount of pollution discharged during the first 3 years of not washing.

$$\text{Total Pollution} = \int_0^3 \frac{1}{t+1}\,dt$$

$$\boxed{\begin{array}{l} u = t+1 \\ \frac{du}{dt} = 1 \\ du = dt \end{array}}$$

$$= \int_1^4 \frac{1}{u}\,du$$

$$= \ln|u|\Big|_1^4$$

$$= [\ln|4|] - [\ln|1|]$$

$$= \ln 4 - 0$$

$$= \ln 4$$

≈ 1.4 liters CALC

George's armpits have sent 1.4 liters of pollution into the air during the first 3 years.