Approximations & Differentials

10.5 – Differential Equations

A differential equation is simply an equation involving derivatives. Many differential equations can be solved by a technique called "separation of variables."

We consider a function y = f(x), which we will sometimes write as y(x) to indicate that y depends on x. We will write the derivative of y as either y' or $\frac{dy}{dx}$.

Differential Equation *y' = f(x)*

We've actually been solving differential equations for a while. For example, the differential equation

$$y' = 2x$$

$y = \int 2x\,dx$

$y = x^2 + C$

General Solution

General and Particular Solutions

The solution of the differential equation $y' = 2x$ is $y = x^2 + C$, with an arbitrary constant C. We call $y = x^2 + C$ the *general solution* because taking all possible values of the constant C gives *all* solutions of the differential equation. If we take C to be a particular number, we get a *particular solution*.

What are some particular solutions to $y' = 2x$?

$y = x^2 + 2$

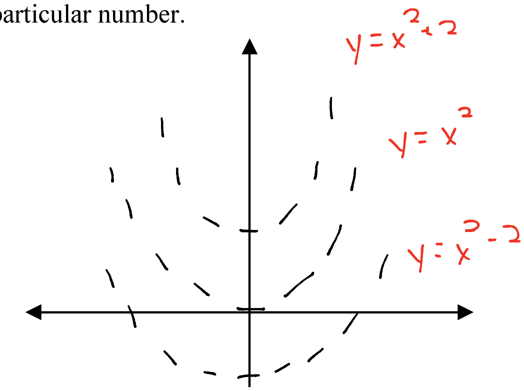
$y = x^2 + 49$

$y = x^2 + e$

$y = x^2 + \pi$

The different values of the arbitrary constant C give a "family" of curves, and the general solution $y = x^2 + C$ may be thought of as the entire family.

The solution of a differential equation is a *function*. The *general* solution contains an arbitrary constant, and a *particular* solution has this constant replaced by a particular number.



Verifying Solutions

Verifying that a function is a solution of a differential equation is simply a matter of calculating the necessary derivatives, substituting them into the differential equation, and checking that the resulting expressions are equal.

10.5 – Differential Equations

#1) Verify that $y = e^{2x} + e^{-x} - 1$ is a solution of the differential equation $y'' - y' - 2y = 2$.

$$y' = 2e^{2x} - e^{-x}$$

$$y'' = 4e^{2x} + e^{-x}$$

$$y'' - y' - 2y = 2$$

$$(4e^{2x} + e^{-x}) - (2e^{2x} - e^{-x}) - 2(e^{2x} + e^{-x} - 1) = 2$$

$$\cancel{4e^{2x}} + e^{-x} \cancel{- 2e^{2x}} + e^{-x} \cancel{- 2e^{2x}} - 2e^{-x} + 2 = 2$$

$$2 = 2$$

#2) Verify that $y = e^{-x} + e^{3x}$ is a solution of the equation $y'' - 2y' - 3y = 0$.

$$y' = -e^{-x} + 3e^{3x}$$

$$y'' = e^{-x} + 9e^{3x}$$

$$y'' - 2y' - 3y = 0$$

$$(e^{-x} + 9e^{3x}) - 2(-e^{-x} + 3e^{3x}) - 3(e^{-x} + e^{3x}) = 0$$

$$e^{-x} + 9e^{3x} + 2e^{-x} - 6e^{3x} - 3e^{-x} - 3e^{3x} = 0$$

$$0 = 0$$

Separable Differential Equations

A first-order differential equation is *separable* if it can be written in the following form for some function *f(x)* and *g(y)*:

If $g(y) \neq 0$

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

Separation of Variables

A differential equation is said to be "separable" if the variables can be "separated" by moving every *x* and *dx* to one side of the equation and every *y* and *dy* to the other side. We may then solve the differential equation by integrating both sides.

#3) Find the general solution of the differential equation $\frac{dy}{dx} = 2xy^2$

$$dy = 2xy^2\,dx$$

$$\frac{dy}{y^2} = 2x\,dx$$

$$\int y^{-2}\,dy = \int 2x\,dx$$

$$-y^{-1} = x^2 + C$$

$$-\frac{1}{y} = x^2 + C$$

$$-1 = y(x^2 + C)$$

$$\frac{-1}{x^2 + C} = y$$

$$y = \frac{-1}{x^2 + C}$$

Approximations & Differentials

10.5 – Differential Equations

Finding a Particular Solution

#4) Solve the differential equation $y' = \frac{6x}{y^2}$ with the initial condition $y(1) = 2$.

$$\frac{dy}{dx} = \frac{6x}{y^2}$$

$$y^2\,dy = 6x\,dx$$

$$\int y^2\,dy = \int 6x\,dx$$

$$\frac{1}{3}y^3 = 3x^2 + C$$

$$y^3 = 9x^2 + 3C$$

Arbitrary C

$$y = \sqrt[3]{9x^2 + C}$$ → General solution

$(1, 2)$

$$2 = \sqrt[3]{9(1)^2 + C}$$

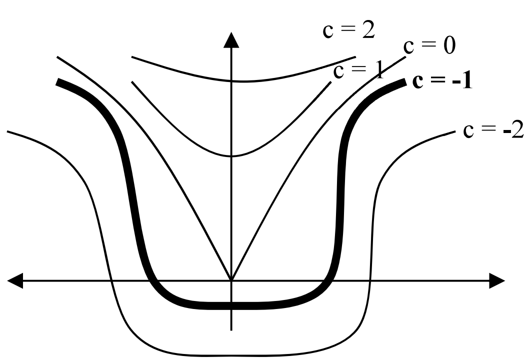
$$2 = \sqrt[3]{9(1) + C}$$

$$2 = \sqrt[3]{9 + C}$$

$$8 = 9 + C$$

$$-1 = C$$

$$y = \sqrt[3]{9x^2 - 1}$$ Particular Solution



#5) Solve the differential equation and initial condition $y' = \frac{6x^2}{y^4}$ if $y(0) = 2$

$$\frac{dy}{dx} = \frac{6x^2}{y^4}$$

$$y^4\,dy = 6x^2\,dx$$

$$\int y^4\,dy = \int 6x^2\,dx$$

$$\frac{1}{5}y^5 = 2x^3 + C$$

$$y^5 = 10x^3 + C$$

$$y = \sqrt[5]{10x^3 + C}$$ ← General

$(0, 2)$

$$2 = \sqrt[5]{10(0)^3 + C}$$

$$(2)^5 = \left(\sqrt[5]{C}\right)^5$$

$$32 = C$$

$$y = \sqrt[5]{10x^3 + 32}$$ ← Particular Solution

#6) Solve the differential equation and initial condition $\frac{dy}{dx} = xy$ if $y(0) = 2$.

$$\frac{dy}{dx} = xy$$

$$dy = xy\,dx$$

$$\frac{dy}{y} = x\,dx$$

$$\int \frac{dy}{y} = \int x\,dx$$

log Form: $\ln|y| = \frac{1}{2}x^2 + C$

Exp Form: $e^{\frac{1}{2}x^2 + C} = y$

$$e^{\frac{1}{2}x^2} e^{C} = y$$

$$C \cdot e^{\frac{1}{2}x^2} = y$$

$y = C \cdot e^{\frac{1}{2}x^2}$ (General Solution)

$(0, 2)$

$$2 = C \cdot e^{\frac{1}{2}(0)^2}$$

$$2 = C e^0$$

$$2 = C$$

$y = 2e^{\frac{1}{2}x^2}$ (Particular Solution)

$\log_a x = y$ | log Form

$a^y = x$ | exp Form

Product Property

$a^{b+c} = a^b \cdot a^c$

Finding a General Solution

#7) Find the general solution of the differential equation $yy' - x = 0$.

$$y \cdot \frac{dy}{dx} = x$$

$$\int y\,dy = \int x\,dx$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$y^2 = x^2 + C$$

$$y = \pm\sqrt{x^2 + C}$$

General Solution w/ Substitution

#8) Solve the differential equation and initial condition
$\boldsymbol{y' = xy - x}$ if $\boldsymbol{y(0) = 4}$

$u = y - 1$
$du = dy$

$$y' = x(y-1)$$
$$\frac{dy}{dx} = x(y-1)$$
$$\int \frac{1}{(y-1)}\,dy = \int x\,dx$$
$$\int \frac{1}{u}\,du = \frac{1}{2}x^2 + C$$
$$\ln|u| = \frac{1}{2}x^2 + C$$
$$\ln|y-1| = \frac{1}{2}x^2 + C$$
$$y - 1 = e^{\frac{1}{2}x^2 + C}$$
$$y = e^{\frac{1}{2}x^2} e^{C} + 1$$
$$y = C \cdot e^{\frac{1}{2}x^2} + 1 \quad \text{(General)}$$

$(0,4)$
$$4 = C \cdot e^{\frac{1}{2}(0)^2} + 1$$
$$4 = C \cdot e^{0} + 1$$
$$3 = C$$

$$y = 3e^{\frac{1}{2}x^2} + 1 \quad \text{(particular solut)}$$

Accumulation of Wealth

The examples so far have *given* us a differential equation to solve. In this application we will first *derive* a differential equation and then solve it.

#9) Suppose that you have saved $5000, and that you expect to save an additional $3000 during each year. If you deposit these savings in a bank account paying 5% interest compounded continuously, find a formula for your bank balance after *t* years.

y = total savings in thousand$
y' = change in $avings per year
t = year

$$y' = \$3 + .05y$$

$$\frac{dy}{dt} = 3 + .05y$$

$$dy = (3 + .05y)\,dt$$

$$\int \frac{dy}{3 + .05y} = \int dt$$

$u = 3 + .05y$
$du = .05\,dy$
$20\,du = dy$

$$\int \frac{20\,du}{u} = t + C$$

$$20 \ln|u| = t + C$$

$$20 \ln|3 + .05y| = t + C$$

$$\ln|3 + .05y| = \frac{t}{20} + C$$

$$e^{\frac{t}{20} + C} = 3 + .05y$$

$$e^{\frac{t}{20}} \cdot e^{C} = 3 + .05y$$

$$C \cdot e^{\frac{t}{20}} = 3 + .05y$$

$$Ce^{\frac{t}{20}} - 3 = .05y$$

$$Ce^{\frac{t}{20}} - 60 = y \quad \text{(General)}$$

$(0, \$5)$

$$5 = Ce^{\frac{0}{20}} - 60$$

$$5 = Ce^{0} - 60$$

$$65 = C$$

$$y = 65e^{\frac{t}{20}} - 60 \quad \text{Particular}$$

17, 21, 25, 29, 33,
37, 41, 45, 49,