Area and Volume
11.2 – Solids of Revolution (Disc Method)

Volume of a Solid of Revolution Disc

$$V = \pi \int_a^b [R(x)]^2 dx$$

Or

$$V = \pi \int_a^b [R(y)]^2 dy$$

Where R(x) or R(y) is the distance between the axis of revolution and the outside of the object.

1) Graph all boundaries

2) Find b (upper bound) and a (lower bound) from visual inspection of graph or by setting boundaries equal to each other.

3) Find $R(x)$
   a. If axis of rotation is x-axis, $R(x) = f(x)$
   b. If axis of rotation is $y = a$, $R(x) = |f(x) - a|$

   OR

   Find $R(y)$
   a. If axis of rotation is y-axis, R(y)=f(y)
   b. If axis of rotation is $x = b$, $R(y) = |f(y) - b|$

4) Find Volume

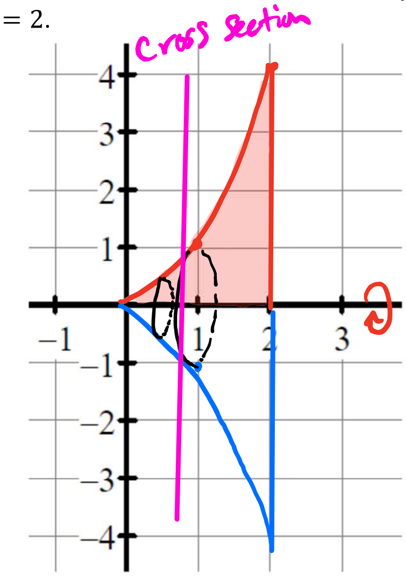
$$Volume = \pi \int_{x=a}^{x=b} [R(x)]^2 dx$$

or

$$Volume = \pi \int_{y=a}^{y=b} [R(y)]^2 dy$$

Finding the volume of a solid of revolution.

1. Sketch the area bounded by the equations. $y = x^2$, $y = 0, x = 2$.



2. Revolve it around the *x*-axis to create a solid.

3. What does the area of a cross section look like?

Circle

4. What is the area of a circle?

$A = \pi r^2$

5. What is the radius of this circle? $f(x)$

Instead of $f(x)$, we will call it $R(x)$.

6. What is the area of one cross-section?

$\pi[x^2]^2$

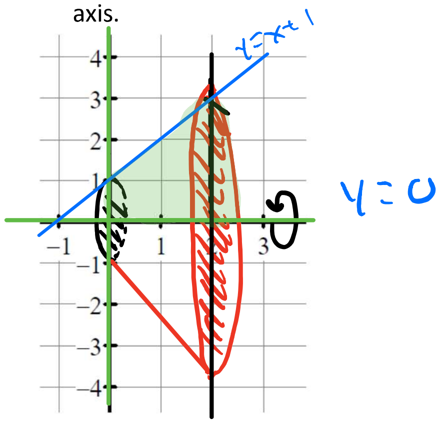
Radius

7. What is the volume of the solid?

$V = \int_0^2 \pi[x^2]^2 dx$

Find the volume of the solid formed by revolving the given boundaries about the given axis of rotation.

1. $y = x + 1, y = 0, x = 0, x = 2$. Revolve about the x-axis.



② $R(x) = x+1$

$R^2 = x^2+2x+1$

③ D: $[0,2]$

④ $V = \pi \int_{x=a}^{x=b} R^2(x)\,dx$

$V = \pi \int_0^2 (x^2+2x+1)\,dx$

$V = \pi \left[\frac{1}{3}x^3+x^2+x\right]\Big|_0^2$

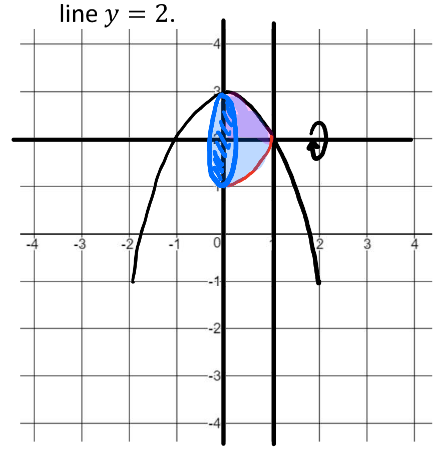
$= \pi\left[\frac{1}{3}(2)^3+(2)^2+(2)\right] - \pi\left[\frac{1}{3}(0)^3+(0)^2+(0)\right]$

$= \pi\left[\frac{8}{3}+4+2\right] - \pi[0]$

$= \pi\left[\frac{8}{3}+\frac{12}{3}+\frac{6}{3}\right]$

$V = \frac{26}{3}\pi$ un^3

2. $y = 3 - x^2, y = 2, x = 0, x = 1$. Revolve about the line $y = 2$.



② $R(x) = (3-x^2)-(2) = -x^2+1$

$R^2 = x^4-2x^2+1$

③ Cross? Not on $(0,1)$; $D=[0,1]$

$$\boxed{\begin{aligned}3-x^2&=2\\1&=x^2\\\pm1&=x\end{aligned}}$$

④ $V = \pi \int_{x=a}^{x=b} R^2(x)\,dx$

$V = \pi \int_0^1 (x^4-2x^2+1)\,dx$

$V = \pi\left[\frac{1}{5}x^5-\frac{2}{3}x^3+x\right]\Big|_0^1$

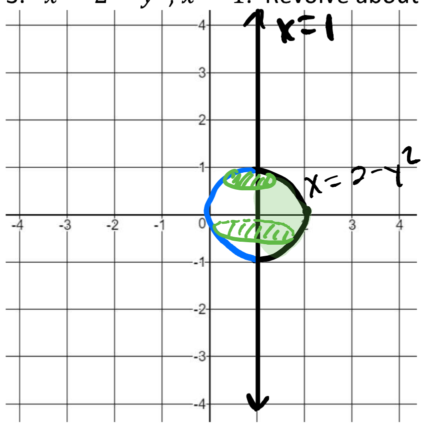
$V = \pi\left[\frac{1}{5}(1)^5-\frac{2}{3}(1)^3+(1)\right] - \pi\left[\frac{1}{5}(0)^5-\frac{2}{3}(0)^3+(0)\right]$

$V = \pi\left[\frac{1}{5}-\frac{2}{3}+1\right]-\pi[0]$

$V = \pi\left[\frac{3}{15}-\frac{10}{15}+\frac{15}{15}\right]$

$V = \frac{8}{15}\pi$ un^3

3. $x = 2 - y^2$, $x = 1$. Revolve about the line $x = 1$.



② $R(x) = (2-y^2) - (1) = -y^2 + 1$

$R^2 = y^4 - 2y^2 + 1$

③ Cross? D: $[-1, 1]$

$$\boxed{\begin{aligned} 2 - y^2 &= 1 \\ 1 &= y^2 \\ \pm 1 &= y \end{aligned}}$$

④ $V = \pi \int_{y=a}^{y=b} R^2(y)\, dy$

$V = \pi \int_{-1}^{1} (y^4 - 2y^2 + 1)\, dy$

$V = \pi \left[\frac{1}{5}y^5 - \frac{2}{3}y^3 + y \right]\Big|_{-1}^{1}$

$= \pi \left[\frac{1}{5}(1)^5 - \frac{2}{3}(1)^3 + (1) \right] - \pi \left[\frac{1}{5}(-1)^5 - \frac{2}{3}(-1)^3 + (-1) \right]$

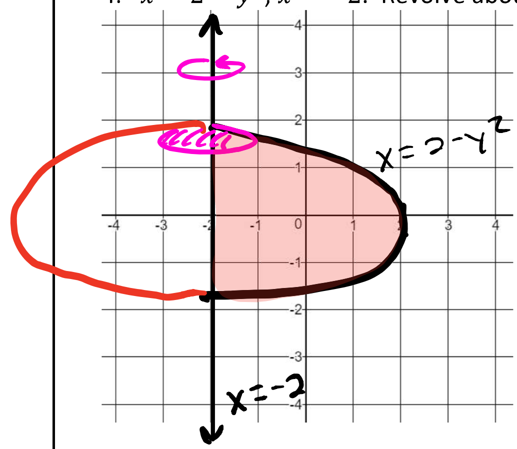
$= \pi \left[\frac{1}{5} - \frac{2}{3} + 1 \right] - \pi \left[-\frac{1}{5} + \frac{2}{3} - 1 \right]$

$= \pi \left[\frac{3}{15} - \frac{10}{15} + \frac{15}{15} \right] - \pi \left[\frac{-3}{15} + \frac{10}{15} - \frac{15}{15} \right]$

$= \pi \left[\frac{8}{15} \right] - \pi \left[\frac{-8}{15} \right]$

$V = \frac{16}{15}\pi \text{ un}^3$

4. $x = 2 - y^2$, $x = -2$. Revolve about the line $x = -2$.



② $R(y) = (2-y^2) - (-2) = -y^2 + 4$

$R^2 = y^4 - 8y^2 + 16$

③ Cross? D: $[-2, 2]$

$$\boxed{\begin{aligned} 2 - y^2 &= -2 \\ 4 &= y^2 \\ \pm 2 &= y \end{aligned}}$$

④ $V = \pi \int_{y=a}^{y=b} R^2(y)\, dy$

$= \pi \int_{-2}^{2} (y^4 - 8y + 16)\, dy$

$= \pi \left[\frac{1}{5}y^5 - 4y^2 + 16y \right]\Big|_{-2}^{2}$

$= \pi \left[\frac{1}{5}(2)^5 - 4(2)^2 + 16(2) \right] - \pi \left[\frac{1}{5}(-2)^5 - 4(-2)^2 + 16(-2) \right]$

$= \pi \left[\frac{32}{5} - 4(4) + 32 \right] - \pi \left[\frac{-32}{5} - 4(4) - 32 \right]$

$= \pi \left[\frac{32}{5} - 16 + 32 \right] - \pi \left[\frac{-32}{5} - 16 - 32 \right]$

$= \pi \left[\frac{32}{5} + 16 \right] - \pi \left[\frac{-32}{5} - 48 \right]$

$= \pi \left[\frac{64}{5} + 64 \right]$

$= \pi \left[\frac{64}{5} + \frac{320}{5} \right]$

$V = \frac{384}{5}\pi \text{ un}^3$