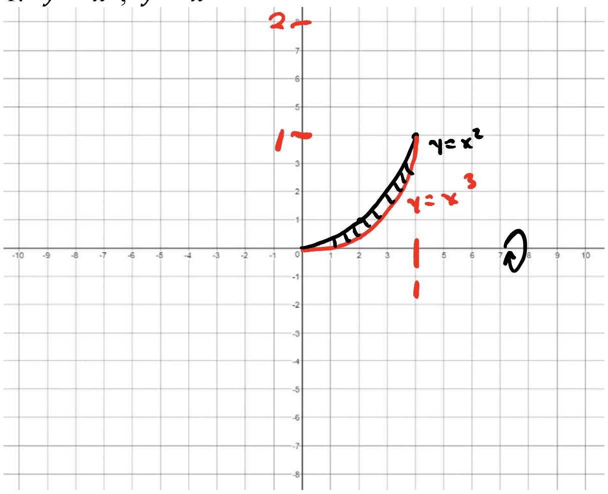
# Area and Volume
## 11.3 – Solids of Revolution Washer Method

Find the volume of the solid formed by revolving it around the x-axis. Leave the answers in terms of $\pi$.

1. $y = x^2,\ y = x^3$




$R(x) = x^2 \Rightarrow R^2 = x^4$
$r(x) = x^3 \Rightarrow r^2 = x^6$

$$V = \pi \int_a^b \left[R^2 - r^2\right] dx$$

$$V = \pi \int_0^1 \left[x^4 - x^6\right] dx$$

$$V = \pi \left[\frac{1}{5}x^5 - \frac{1}{7}x^7\right]\Big|_0^1$$

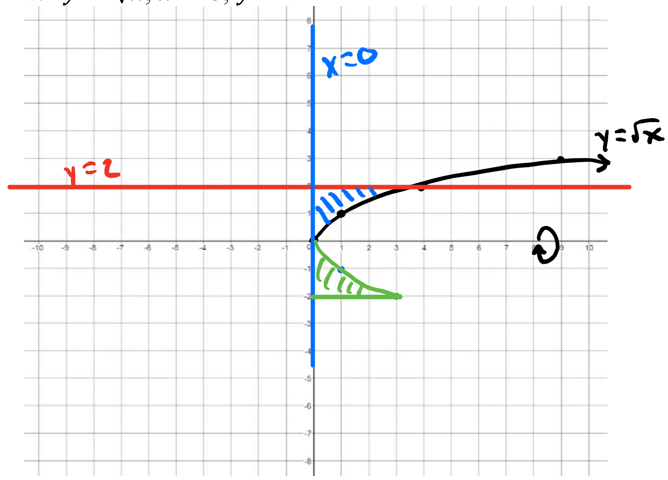
$$V = \pi\left[\frac{1}{5}(1)^5 - \frac{1}{7}(1)^7\right] - \pi\left[\frac{1}{5}(0)^5 - \frac{1}{7}(0)^7\right]$$

$$V = \pi\left[\frac{1}{5} - \frac{1}{7}\right] - \pi[0]$$

$$V = \pi\left[\frac{7}{35} - \frac{5}{35}\right]$$

$$V = \frac{2}{35}\pi \text{ un}^3$$

2. $y = \sqrt{x},\ x = 0,\ y = 2$




$R(x) = 2 \Rightarrow R^2 = 4$
$r(x) = \sqrt{x} \Rightarrow r^2 = x$

$$V = \pi \int_a^b \left[R^2 - r^2\right] dx$$

$$V = \pi \int_0^4 \left[4 - x\right] dx$$

$$V = \pi\left[4x - \frac{1}{2}x^2\right]\Big|_0^4$$

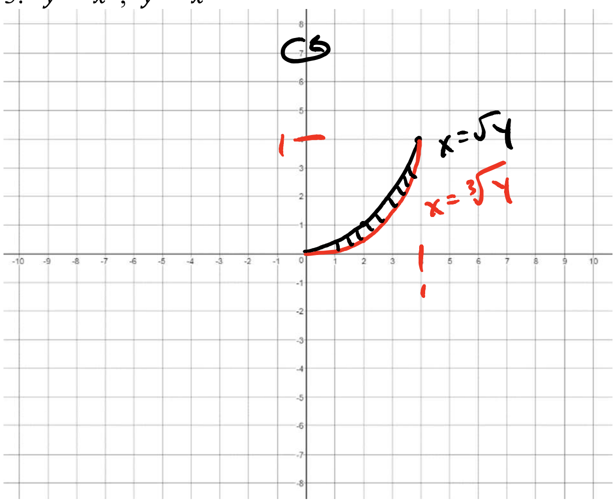
$$V = \pi\left[4(4) - \frac{1}{2}(4)^2\right] - \pi\left[4(0) - \frac{1}{2}(0)^2\right]$$

$$V = \pi[16 - 8] - \pi[0]$$

$$V = 8\pi \text{ un}^3$$

Find the volume of the solid formed by revolving it around the y-axis. Leave the answers in terms of $\pi$.

3. $y = x^2,\ y = x^3$




$R(y) = \sqrt[3]{y} \Rightarrow R^2 = y^{2/3}$

$r(y) = \sqrt{y} \Rightarrow r^2 = y$

$$V = \pi \int_a^b [R^2 - r^2]\,dy$$

$$V = \pi \int_0^1 [y^{2/3} - y]\,dy$$

$$V = \pi \left[\tfrac{3}{5} y^{5/3} - \tfrac{1}{2} y^2\right]\Big|_0^1$$

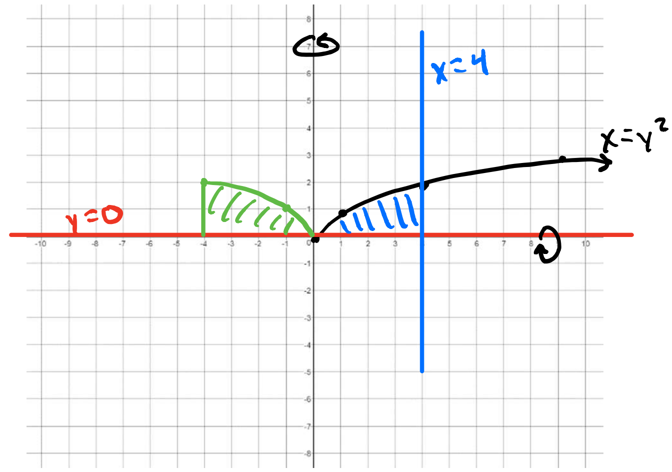
$$V = \pi \left[\tfrac{3}{5}(1)^{5/3} - \tfrac{1}{2}(1)^2\right] - \pi\left[\tfrac{3}{5}(0)^{5/3} - \tfrac{1}{2}(0)^2\right]$$

$$V = \pi\left[\tfrac{3}{5} - \tfrac{1}{2}\right] - \pi[0]$$

$$V = \pi\left[\tfrac{6}{10} - \tfrac{5}{10}\right]$$

$$V = \tfrac{1}{10}\pi \text{ un}^3$$

4. $y = \sqrt{x},\ y = 0,\ x = 4$




$R(x) = 4 \Rightarrow R^2 = 16$

$r(x) = y^2 \Rightarrow r^2 = y^4$

$$V = \pi \int_a^b [R^2 - r^2]\,dy$$

$$V = \pi \int_0^2 [16 - y^4]\,dy$$

$$V = \pi \left[16y - \tfrac{1}{5} y^5\right]\Big|_0^2$$

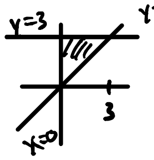
$$V = \pi\left[16(2) - \tfrac{1}{5}(2)^5\right] - \pi\left[16(0) - \tfrac{1}{5}(0)^5\right]$$

$$V = \pi\left[32 - \tfrac{32}{5}\right] - \pi[0]$$

$$V = \pi\left[\tfrac{160}{5} - \tfrac{32}{5}\right]$$

$$V = \tfrac{128}{5}\pi \text{ un}^3$$

5. Sketch the graph and find the area of the region bounded by $y = x, x = 0, y = 3$




$A = \int_0^3 (3-x)\,dx = \left[3x - \frac{1}{2}x^2\right]_0^3$

$= \left[3(3) - \frac{1}{2}(3)^2\right] - \left[3(0) - \frac{1}{2}(0)^2\right]$

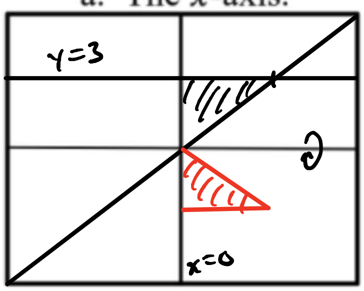
$= \left[9 - \frac{9}{2}\right] - [0]$

$= \frac{18}{2} - \frac{9}{2}$

$A = \frac{9}{2}$ un²

Set up the integral to find the volume when revolving it about the given line. DO NOT EVALUATE!

a. The $x$-axis.

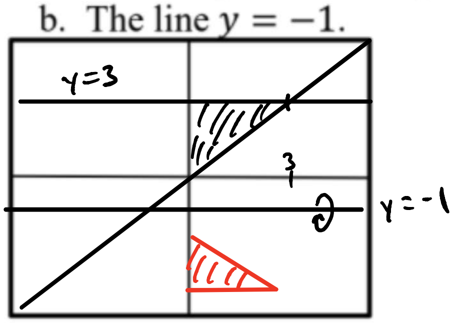



$R = 3 \Rightarrow R^2 = 9$

$r = x \Rightarrow r^2 = x^2$

$V = \pi \int_0^3 \left[9 - x^2\right] dx$

b. The line $y = -1$.

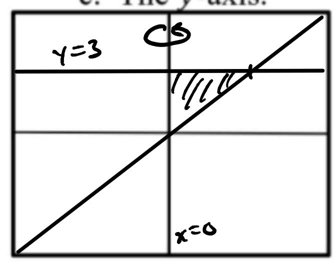



$R = 3 - (-1) = 4 \Rightarrow R^2 = 16$

$r = x - (-1) = x + 1 \Rightarrow r^2 = (x+1)^2$

$V = \pi \int_0^3 \left[16 - (x+1)^2\right] dx$

c. The $y$-axis.

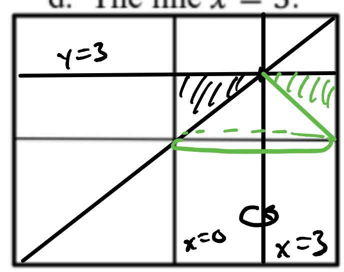



$R = y \Rightarrow R^2 = y^2$

$r = 0 \Rightarrow r = 0$

$V = \pi \int_0^3 y^2\,dy$

d. The line $x = 3$.

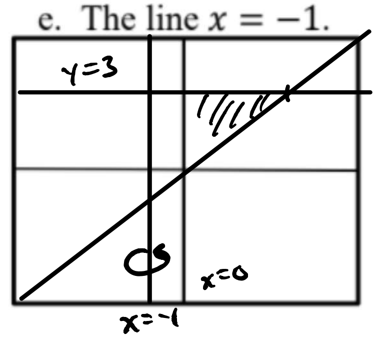



$R = 3 - 0 = 3 \Rightarrow R^2 = 9$

$r = 3 - y \Rightarrow r = (3-y)^2$

$V = \pi \int_0^3 \left[9 - (3-y)^2\right] dy$

e. The line $x = -1$.



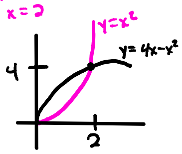

$R = y - (-1) = y + 1 \Rightarrow R^2 = (y+1)^2$

$r = 0 - (-1) = 1 \Rightarrow r^2 = 1$

$V = \pi \int_0^3 \left[(y+1)^2 - 1\right] dy$

6. Sketch the graph and find the area of the region bounded by $y = x^2, y = 4x - x^2$

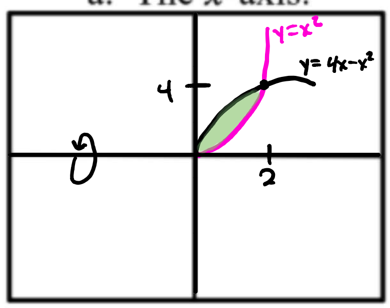
Cross?
$x^2 = 4x - x^2$
$2x^2 - 4x = 0$
$2x(x-2) = 0$
$2x = 0$, $x = 0$; $x - 2 = 0$, $x = 2$




$$A = \int_0^2 [(4x-x^2)-x^2]\,dx = \int_0^2 [4x-2x^2]\,dx$$
$$= \left[2x^2 - \tfrac{2}{3}x^3\right]_0^2$$
$$= \left[2(2)^2 - \tfrac{2}{3}(2)^3\right] - \left[2(0)^2 - \tfrac{2}{3}(0)^3\right]$$
$$= \left[8 - \tfrac{16}{3}\right] - [0]$$
$$= \tfrac{24}{3} - \tfrac{16}{3}$$
$$A = 8/3 \text{ un}^2$$

Set up the integral to find the volume when revolving it about the given line. DO NOT EVALUATE!

a. The $x$-axis.

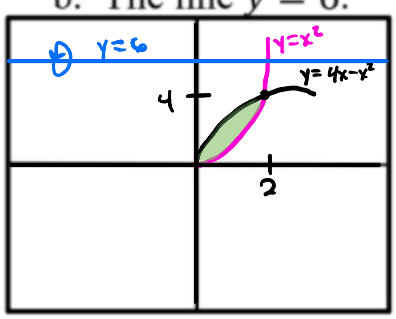



$R = 4x - x^2 \Rightarrow R^2 = (4x-x^2)^2$

$r = x^2 \Rightarrow r^2 = x^4$

$V = \pi \int_0^2 [(4x-x^2)^2 - x^4]\,dx$

b. The line $y = 6$.



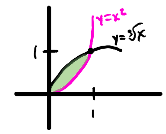

$R = 6 - x^2 \Rightarrow R^2 = (6-x^2)^2$

$r = 6-(4x-x^2) = 6-4x+x^2 \Rightarrow r^2 = (6-4x+x^2)^2$

$V = \pi \int_0^2 [(6-x^2)^2 - (6-4x+x^2)^2]\,dx$

7. Sketch the graph and find the area of the region bounded by $y = x^2, y = \sqrt[3]{x}$

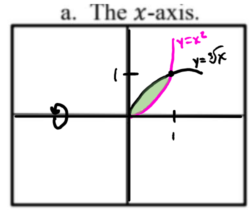
Cross?
$x^2 = \sqrt[3]{x}$
$x^6 = x$
$x^6 - x = 0$
$x(x^5-1) = 0$
$x = 0$; $x^5 - 1 = 0$, $x^5 = 1$, $x = 1$




$$A = \int_0^1 (x^{1/3} - x^2)\,dx = \left[\tfrac{3}{4}x^{4/3} - \tfrac{1}{3}x^3\right]_0^1$$
$$= \left[\tfrac{3}{4}(1)^{4/3} - \tfrac{1}{3}(1)^3\right] - \left[\tfrac{3}{4}(0)^{4/3} - \tfrac{1}{3}(0)^3\right]$$
$$= \left[\tfrac{3}{4} - \tfrac{1}{3}\right] - [0]$$
$$= \tfrac{9}{12} - \tfrac{4}{12}$$
$$A = \tfrac{5}{12} \text{ un}^2$$

Set up the integral to find the volume when revolving it about the given line. DO NOT EVALUATE!

a. The $x$-axis.

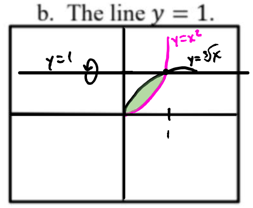



$R = \sqrt[3]{x} \Rightarrow R^2 = x^{2/3}$

$r = x^2 \Rightarrow r^2 = x^4$

$V = \pi \int_0^1 [x^{2/3} - x^4]\,dx$

b. The line $y = 1$.

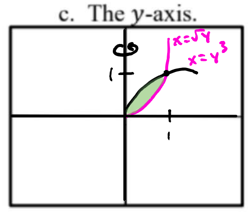



$R = 1 - x^2 \Rightarrow R^2 = (1-x^2)^2$

$r = 1 - \sqrt[3]{x} \Rightarrow r^2 = (1-\sqrt[3]{x})^2$

$V = \pi \int_0^1 [(1-x^2)^2 - (1-\sqrt[3]{x})^2]\,dx$

c. The $y$-axis.

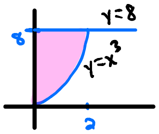



$R = \sqrt{y} \Rightarrow R^2 = y$

$r = y^3 \Rightarrow r^2 = y^6$

$V = \pi \int_0^1 [y - y^6]\,dy$

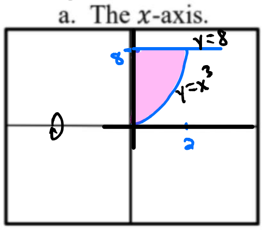
8. Sketch the graph and find the area of the region bounded by $y = x^3, x = 0, y = 8$




$$A = \int_0^2 (8 - x^3)\,dx = \left[8x - \tfrac{1}{4}x^4\right]\Big|_0^2$$
$$= \left[8(2) - \tfrac{1}{4}(2)^4\right] - \left[8(0) - \tfrac{1}{4}(0)^4\right]$$
$$= [16 - 4] - [0]$$
$$A = 12 \text{ un}^2$$

Set up the integral to find the volume when revolving it about the given line. DO NOT EVALUATE!

a. The $x$-axis.

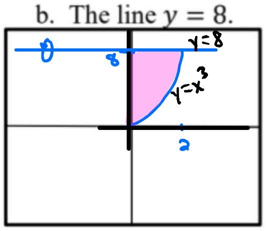



$R = 8 \Rightarrow R^2 = 64$

$r = x^3 \Rightarrow r^2 = x^6$

$V = \pi \int_0^2 [64 - x^6]\,dx$

b. The line $y = 8$.

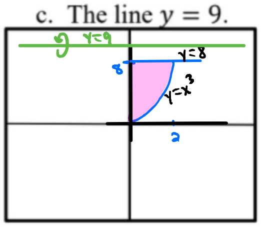



$R = 8 - x^3 \Rightarrow R^2 = (8 - x^3)^2$

$r = 8 - 8 = 0 \Rightarrow r^2 = 0$

$V = \pi \int_0^2 (8 - x^3)^2\,dx$

c. The line $y = 9$.

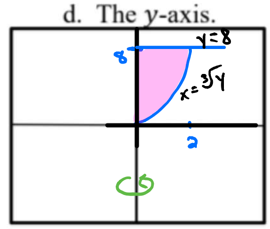



$R = 9 - x^3 \Rightarrow R^2 = (9 - x^3)^2$

$r = 9 - 8 = 1 \Rightarrow r^2 = 1$

$V = \pi \int_0^2 [(9 - x^3)^2 - 1]\,dx$

d. The $y$-axis.

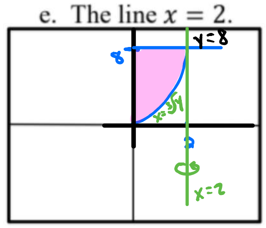



$R = \sqrt[3]{y} \Rightarrow R^2 = y^{2/3}$

$r = 0 \Rightarrow r^2 = 0$

$V = \pi \int_0^8 y^{2/3}\,dy$

e. The line $x = 2$.

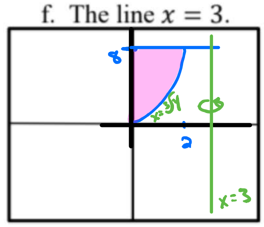



$R = 2 \Rightarrow R^2 = 4$

$r = 2 - \sqrt[3]{y} \Rightarrow r^2 = (2 - \sqrt[3]{y})^2$

$V = \pi \int_0^8 [4 - (2 - \sqrt[3]{y})^2]\,dy$

f. The line $x = 3$.




$R = 3 \Rightarrow R^2 = 9$

$r = 3 - \sqrt[3]{y} \Rightarrow r^2 = (3 - \sqrt[3]{y})^2$

$V = \pi \int_0^8 [9 - (3 - \sqrt[3]{y})^2]\,dy$

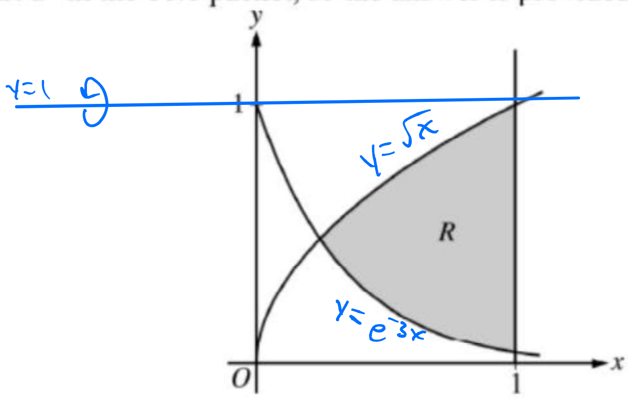
## 11.3 Solids of Revolution (Washers) **Test Prep**

1. The area bounded by the curves $y = x^2 + 4$ and $y = -2x + 1$ between $x = -2$ and $x = 5$ equals

(A) 86.500 (B) 86.425 (C) 86.333 (D) 86.125 (E) 86.000

**2003 Form A #1 [calculator allowed]**

You already did "part a" in the 11.1 packet, so the answer is provided for you. Now do part b.




Let $R$ be the shaded region bounded by the graphs of $y = \sqrt{x}$ and $y = e^{-3x}$ and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area $R$.

$(x, y)$

Point of intersection: $e^{-3x} = \sqrt{x}$ at $(0.238734,\ 0.488604)$

Area $= \int_{0.238734}^{1} \left(\sqrt{x} - e^{-3x}\right) dx$

(b) Find the volume of the solid generated when $R$ is revolved about the horizontal line $y = 1$.

$R = 1 - e^{-3x} \Rightarrow R^2 = (1 - e^{-3x})^2 = 1 - 2e^{-3x} + e^{-6x}$

$r = 1 - \sqrt{x} \Rightarrow r^2 = (1 - \sqrt{x})^2 = 1 - 2\sqrt{x} + x$

$V = \pi \int_{0.238734}^{1} \left[(1 - 2e^{-3x} + e^{-6x}) - (1 - 2\sqrt{x} + x)\right] dx$

$V = \pi \int_{0.238734}^{1} \left[-2e^{-3x} + e^{-6x} + 2x^{\frac{1}{2}} + x\right] dx$

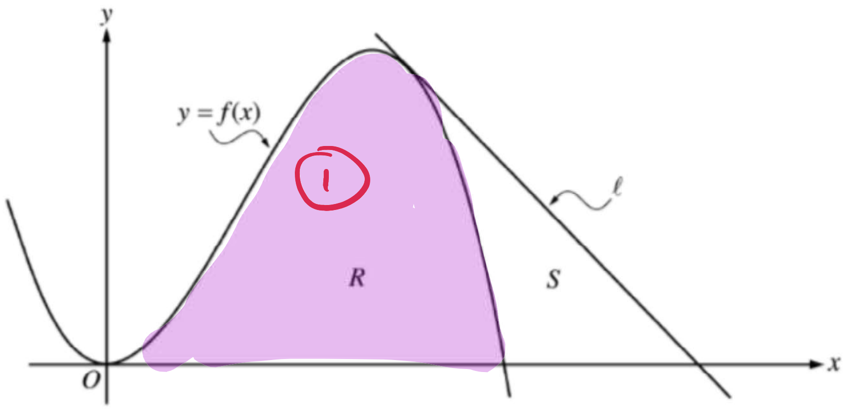
$V = \pi \left[\frac{2}{3}e^{-3x} - \frac{1}{6}e^{-6x} + \frac{4}{3}x^{3/2} + \frac{1}{2}x^2\right]\Big|_{0.238734}^{1}$

$V = \pi\left[\frac{2}{3}e^{-3(1)} - \frac{1}{6}e^{-6(1)} + \frac{4}{3}(1)^{3/2} + \frac{1}{2}(1)^2\right] - \pi\left[\frac{2}{3}e^{-3(0.238734)} - \frac{1}{6}e^{-6(0.238734)} + \frac{4}{3}(0.238734)^{3/2} + \frac{1}{2}(0.238734)^2\right]$

$V = 0.453\pi$

**2003 Form B #1 [calculator allowed]**

You already did parts "a" and "b" in the 11.1 packet.




Let $f$ be the function given by $f(x) = 4x^2 - x^3$, and let $\ell$ be the line $y = 18 - 3x$, where $\ell$ is tangent to the graph of $f$. Let $R$ be the region bounded by the graph of $f$ and the $x$-axis, and let $S$ be the region bounded by the graph of $f$, the line l, and the $x$-axis, as shown above.

(c) Find the volume of the solid generated when $R$ is revolved about the $x$-axis.

2 $R = 4x^2 - x^3$

$R^2 = (4x^2 - x^3)^2$

3 $D = [0, 4]$

$4x^2 - x^3 = 0$

$x^2(4 - x) = 0$

$x^2 = 0$; $4 - x = 0$

$x = 0$ $\quad$ $x = 4$

4 $V = \pi \int_a^b R^2 \, dx$

$= \pi \int_0^4 (4x^2 - x^3)^2 dx$

$V \approx 156.038\pi$ un$^3$ (CALC)

2003 Form B #1

(c) Volume $= \pi \int_0^4 \left(4x^2 - x^3\right)^2 dx$

$= 156.038\pi$ or $490.208$

3 : { 1 : limits and constant
1 : integrand
1 : answer