Lagrange ′
Newton ˙
Leibniz $\frac{d}{dx}$

Notes 3.1 – Rules for Differentiation
Finding the Derivative of a Product of Two Functions

Rewrite the function $f(x) = (2x - 3)(x^2 - 2x + 1)$ as a cubic function. Then, find $f'(x)$. What does this equation of $f'(x)$ represent, again?

$f(x) = 2x^3 - 4x^2 + 2x - 3x^2 + 6x - 3$
$f(x) = 2x^3 - 7x^2 + 8x - 3$
$\dot{f}(x) = 6x^2 - 14x + 8$

$\dot{f}(x)$ represent the slopes of all the tangent lines of $f(x)$

Two men, Isaac Newton and Gottfried Leibniz, are credited for developing the study of calculus. In 1673, Leibniz published an article in which he derived what we know today as the Product Rule of Differentiation. Let's write this rule together in the box below.

Product Rule of Differentiation

If $h(x) = f(x) \cdot g(x)$, then $\frac{d}{dx} h(x) = \frac{d}{dx} f(x) \cdot g(x) + f(x) \cdot \frac{d}{dx} g(x)$

To show that this rule works, let's apply this rule to the function $f(x) = (2x - 3)(x^2 - 2x + 1)$ that we rewrote and differentiated as a polynomial above.

$\frac{d}{dx} f(x) = \frac{d}{dx}(2x-3) \cdot (x^2 - 2x + 1) + (2x-3) \cdot \frac{d}{dx}(x^2 - 2x + 1)$
$= 2 \cdot (x^2 - 2x + 1) + (2x-3)(2x-2)$
$= 2x^2 - 4x + 2 + 4x^2 - 4x - 6x + 6$
$\frac{d}{dx} f(x) = 6x^2 - 14x + 8$

Students often wonder why this rule is so important if we could just rewrite as a polynomial and easily differentiate it. The answer to that question is simple. If it is possible to rewrite as a polynomial, always do so. But in the case of the function $g(x) = x^2 \sin x$, there is no way to rewrite as a polynomial. Apply the product rule to find the slope of the normal line to the graph of $g(x) = x^2 \sin x$ when $x = \pi$.

SON : $m = \frac{1}{\pi^2}$

$g'(x) = 2x \cdot \sin x + x^2 (\cos x)$
$g'(\pi) = 2\pi \cdot \sin \pi + (\pi)^2 \cos \pi$
$= 2\pi (0) + \pi^2 (-1)$
$= 0 + (-\pi)^2$
$g'(\pi) = -\pi^2$ (SOT)

The slope of normal line is $\frac{1}{\pi^2}$ when $x = \pi$.

Use the product rule to find the derivative of each of the following functions.

$f(x) = (2x^2+3x)(x^2-3)$

$f' = (4x+3)(x^2-3) + (2x^2+3x)\cdot 2x$

$f' = 4x^3 - 12x + 3x^2 - 9 + 4x^3 + 6x^2$

$f' = 8x^3 + 9x^2 - 12x - 9$

$g(x) = \sqrt{x}(x^2-3x+2)$ $\sqrt{x} \to x^{\frac{1}{2}}$

$g' = \frac{1}{2}x^{-\frac{1}{2}}\cdot(x^2-3x+2) + x^{\frac{1}{2}}(2x-3)$

$= x^{-\frac{1}{2}}\left[\frac{1}{2}(x^2-3x+2) + x^1(2x-3)\right]$

$= \dfrac{\frac{1}{2}x^2 - \frac{3}{2}x + 1 + 2x^2 - 3x}{\sqrt{x}}$

$= \dfrac{\frac{5}{2}x^2 - \frac{9}{2}x + 1}{\sqrt{x}}$

$f(x) = x^3 \sin x$

$\dot{f} = 3x^2\cdot \sin x + x^3 \cdot \cos x$

$\dot{f} = x^2(3\sin x + x\cos x)$

$h(x) = (3x+2)\cos x$

$\dot{h}(x) = 3\cdot \cos x + (3x+2)(-\sin x)$

$\dot{h}(x) = 3\cos x - 3x\sin x - 2\sin x$

$g(\theta) = 3\theta + \theta\sin\theta$

$g'(\theta) = (3\theta)' + (\theta\cdot\sin\theta)'$

$= 3 + \theta'\cdot\sin\theta + \theta\cdot(\sin\theta)'$

$\frac{d}{d\theta}g(\theta) = 3 + \underbrace{1\cdot\sin\theta + \theta\cdot\cos\theta}_{\text{Product Rule}}$

$\frac{dg}{d\theta} = 3 + \sin\theta + \theta\cos\theta$

$h(x) = \sin x \cos x$

$h'(x) = \cos x\cdot\cos x + \sin x(-\sin x)$

$= \cos^2 x - \sin^2 x$

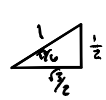
Find the equation of the line tangent to the graph of $g(t) = t^2 \cos t$ when $t = \frac{\pi}{6}$.

PoT $\left(\frac{\pi}{6}, \frac{\pi^2\sqrt{3}}{72}\right)$

$g\left(\frac{\pi}{6}\right) = \left(\frac{\pi}{6}\right)^2\cos\left(\frac{\pi}{6}\right)$

$= \frac{\pi^2}{36}\cdot\frac{\sqrt{3}}{2}$

$g\left(\frac{\pi}{6}\right) = \frac{\pi^2\sqrt{3}}{72}$



SoT: $m = \frac{12\sqrt{3}\pi - \pi^2}{72}$

$g'(t) = 2t\cdot\cos t + t^2\cdot(-\sin t)$

$g'(t) = 2t\cos t - t^2\sin t$

$g'\left(\frac{\pi}{6}\right) = 2\cdot\frac{\pi}{6}\cos\frac{\pi}{6} - \left(\frac{\pi}{6}\right)^2\sin\frac{\pi}{6}$

$= \frac{\pi}{3}\cdot\frac{\sqrt{3}}{2} - \frac{\pi^2}{36}\cdot\frac{1}{2}$

$= \frac{12\cdot\sqrt{3}\pi}{12\cdot 6} - \frac{\pi^2}{72}$

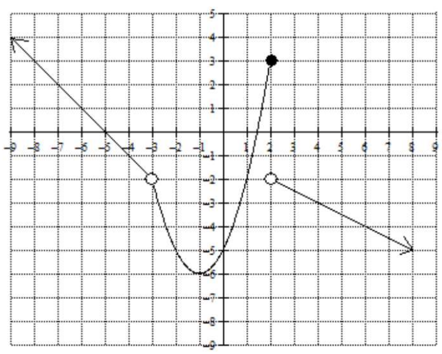
$g'\left(\frac{\pi}{6}\right) = \frac{12\sqrt{3}\pi - \pi^2}{72}$

Tangent line

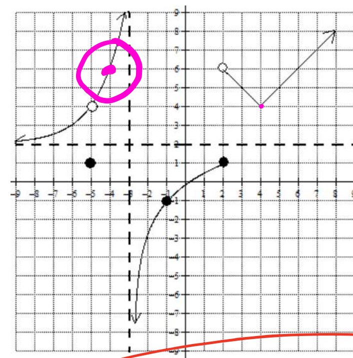
$y - \frac{\pi^2\sqrt{3}}{72} = \frac{12\sqrt{3}\pi - \pi^2}{72}\left(x - \frac{\pi}{6}\right)$

Below are graphs of two functions—$f(x)$ and $g(x)$. Let $P(x) = f(x)\cdot g(x)$ and let $R(x) = x^2 \cdot g(x)$. Use the graphs to answer the questions that follow.

Graph of $f(x)$



Graph of $g(x)$



If $g'(-4) = 2$, what is the value of $P'(-4)$?

$P(x) = f(x)\cdot g(x)$

$P'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

$P'(-4) = f'(-4)\cdot g(-4) + f(-4)\cdot g'(-4)$

$= (-1)\cdot 6 + (-1)\cdot 2$

$= -6 + (-2)$

$P'(-4) = -8$

If $R'(-2) = 20$, what is the value of $g'(-2)$?

$R(x) = x^2\cdot g(x)$

$R'(x) = 2x\cdot g(x) + x^2\cdot g'(x)$

$R'(-2) = 2(-2)\cdot g(-2) + (-2)^2\cdot g'(-2)$

$20 = -4\cdot(-3) +$ [illegible] $\cdot g'(-2)$

$20 = 12 + 4g'(-2)$

$8 = 4\cdot g'(-2)$

$2 = g'(-2)$

Find the equation of the line tangent to the graph of $P(x)$ when $x = -4$.

PoT: $(-4, -6)$

SoT

$m = -8$

$P(-4) = f(-4)\cdot g(-4)$

$= (-1)(6)$

$P(-4) = -6$

Tan line

$y + 6 = -8(x+4)$

Find the equation of the line tangent to the graph of $R(x)$ when $x = -2$.

PoT: $(-2, -12)$

SoT m: 20

$R(x) = x^2\cdot g(x)$

$R(-2) = (-2)^2\cdot g(-2)$

$= 4\cdot(-3)$

$R(-2) = -12$

$y + 12 = 20(x+2)$

Let $f(x)$ and $g(x)$ be differentiable functions such that the following values are true.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
4	1	7	2	–3
3	–2	–3	–4	2
–1	2	–2	1	–1

$m = \frac{\Delta y}{\Delta x}$

Estimate the value of $f'(3.5)$.

$f'(3.5) \approx \frac{f(4)-f(3)}{4-3} = \frac{1-(-2)}{1} = 3/1$

$f'(3.5) \approx 3$

If $q(x) = 2f(x) - 4g(x)$, what is the value of $q'(4)$?

$q'(x) = 2f'(x) - 4g'(x)$

$q'(4) = 2 \cdot f'(4) - 4g'(4)$

$= 2 \cdot (2) - 4 \cdot (-3)$

$= 4 + 12$

$q'(4) = 16$

If $p(x) = -2f(x)g(x)$, what is the value of $p'(3)$?

$p'(x) = -2f'(x) \cdot g(x) - 2f(x) \cdot g'(x)$

$p'(3) = -2f'(3) \cdot g(3) - 2f(3) \cdot g'(3)$

$= -2 \cdot (-4) \cdot (-3) - 2 \cdot (-2) \cdot (2)$

$= -24 + 8$

$p(3) = -16$

Find the equation of the line tangent to the graph of $v(x) = x^3 \cdot f(x)$ when $x = -1$.

PoT $(-1, -2)$

$v(-1) = (-1)^3 \cdot f(-1)$

$= -1 \cdot 2$

$v(-1) = -2$

SoT: $m = 5$

$v'(x) = 3x^2 \cdot f(x) + x^3 \cdot f'(x)$

$v'(-1) = 3(-1)^2 \cdot f(-1) + (-1)^3 \cdot f'(-1)$

$= 3(1) \cdot 2 + (-1)(1)$

$= 6 - 1$

$v'(-1) = 5$

Tangent Line

$y + 2 = 5(x + 1)$

If $k(x) = (2f(x) + 3)(3 - g(x))$, what is the value of $k'(3)$?

$k'(x) = [2 \cdot f'(x)][3 - g(x)] + [2f(x) + 3][-g'(x)]$

$k'(3) = 2 \cdot f'(3)[3 - g(3)] + [2 \cdot f(3) + 3][-g'(3)]$

$= 2 \cdot (-4)[3 - (-3)] + [2 \cdot (-2) + 3][-(2)]$

$= -8(6) + [-1][-2]$

$= -48 + 2$

$k'(3) = -46$