Notes 3.2 – Rules for Differentiation

Finding the Derivative of a Quotient of Two Functions

Rewrite the function $f(x)=\dfrac{2x^3-3x^2+2}{x^2}$ as a function in polynomial form. Then, find $f'(x)$.

$f(x)=\dfrac{2x^3}{x^2}-\dfrac{3x^2}{x^2}+\dfrac{2}{x^2}$

$f(x)=2x-3+2x^{-2}$

$f'(x)=2-4x^{-3}$

$=\dfrac{2x^3}{x^3}-\dfrac{4}{x^3}$

$f'(x)=\dfrac{2x^3-4}{x^3}$

Just as Leibniz was the first to publish a proof of the Product Rule for Differentiation, Isaac Newton was the first to publish a proof of the Quotient Rule of Differentiation using the limit definition of the derivative. Let's write this rule together in the box below.

Quotient Rule of Differentiation

If $h(x)=\dfrac{f(x)}{g(x)}$, then $h'(x)=\dfrac{f'\cdot g-f\cdot g'}{g^2}$

To show that this rule works, let's apply this rule to the function $f(x)=\dfrac{2x^3-3x^2+2}{x^2}$ that we rewrote and differentiated as a polynomial-form above.

$f'=\dfrac{(6x^2-6x)\cdot x^2-(2x^3-3x^2+2)\cdot 2x}{(x^2)^2}$

$f'=\dfrac{6x^4\cancel{-6x^3}-4x^4\cancel{+6x^3}-4x}{x^4}$

$f'=\dfrac{2x^4-4x}{x^4}$

$f'=\dfrac{x(2x^3-4)}{x^4}\Rightarrow f'=\dfrac{2x^3-4}{x^3}$

Find the equation of the tangent line drawn to the graph of $g(x)=\dfrac{2x-1}{x+3}$ when $x=-2$.

PoT $(-2,-5)$

$g(-2)=\dfrac{2(-2)-1}{(-2)+3}$

$=\dfrac{-4-1}{1}$

$=\dfrac{-5}{1}$

$g(-2)=-5$

SoT: $m=7$

$g'(x)=\dfrac{(2)(x+3)-(2x-1)(1)}{(x+3)^2}$

$g'(x)=\dfrac{2x+6-2x+1}{(x+3)^2}$

$g'(x)=\dfrac{7}{(x+3)^2}$

$g'(-2)=\dfrac{7}{[(-2)+3]^2}$

$=\dfrac{7}{(1)^2}$

$g'(-2)=7$

Tangent Line

$y+5=7(x+2)$

Show, using the quotient rule, that if $f(x) = \dfrac{x^2+3x+2}{x^2-1}$, then $f'(x) = -\dfrac{3}{(x-1)^2}$.

$$f'(x) = \frac{(2x+3)(x^2-1) - (x^2+3x+2)(2x)}{(x^2-1)^2}$$

$$f'(x) = \frac{\cancel{2x^3} - 2x + 3x^2 - 3 - \cancel{2x^3} - 6x^2 - 4x}{(x^2-1)^2}$$

$$= \frac{-3x^2 - 6x - 3}{(x^2-1)^2}$$

$$= \frac{-3(x^2+2x+1)}{[(x-1)(x+1)]^2}$$

$$= \frac{-3\cancel{(x+1)}\cancel{(x+1)}}{(x-1)\cancel{(x+1)}(x-1)\cancel{(x+1)}}$$

$$f'(x) = \frac{-3}{(x-1)^2}$$

Similar to the Product Rule, there is a very valuable lesson that we must learn when we are introduced to the quotient rule. In the box below, first factor and simplify the function, $f(x) = \dfrac{x^2+3x+2}{x^2-1}$, from above. Then, differentiate using the quotient rule

$$f(x) = \frac{x^2+3x+2}{x^2-1}$$

$$= \frac{\cancel{(x+1)}(x+2)}{\cancel{(x+1)}(x-1)}$$

$$f(x) = \frac{x+2}{x-1}$$

$$\dot{f} = \frac{(1)(x-1) - (x+2)(1)}{(x-1)^2}$$

$$= \frac{x-1-x-2}{(x-1)^2}$$

$$\dot{f} = \frac{-3}{(x-1)^2}$$

What is the lesson to be learned from the algebraic analysis above?

guess:

We will now use the quotient rule to derive the derivative formulas for the remaining trigonometric functions. Rewrite each function in terms of sine and/or cosine and differentiate using the Quotient Rule.

$f(\theta) = \tan\theta = \frac{\sin\theta}{\cos\theta}$

$$f'(\theta) = \frac{(\cos\theta)(\cos\theta) - (\sin\theta)(-\sin\theta)}{(\cos\theta)^2}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta}$$

$$f' = \sec^2\theta$$

$$\therefore (\tan\theta)' = \sec^2\theta$$

$f(\theta) = \cot\theta = \frac{\cos\theta}{\sin\theta}$

$$f' = \frac{(-\sin\theta)(\sin\theta) - (\cos\theta)(\cos\theta)}{\sin^2\theta}$$

$$= \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta}$$

$$= \frac{-1(\sin^2\theta + \cos^2\theta)}{\sin^2\theta}$$

$$= \frac{-1}{\sin^2\theta}$$

$$f' = -\csc^2\theta$$

$$\therefore \cot'\theta = -\csc^2\theta$$

$f(\theta) = \sec\theta = \frac{1}{\cos\theta}$

$$f' = \frac{(0)\cdot\cos\theta - 1\cdot(-\sin\theta)}{\cos^2\theta}$$

$$= \frac{\sin\theta}{\cos^2\theta}$$

$$= \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}$$

$$f' = \sec\theta\tan\theta$$

$$\therefore \frac{d}{d\theta}[\sec\theta] = \sec\theta\tan\theta$$

$f(\theta) = \csc\theta = \frac{1}{\sin\theta}$

$$\frac{d}{d\theta}[\csc\theta] = \frac{(0)\cdot\sin\theta - 1\cdot(\cos\theta)}{\sin^2\theta}$$

$$= \frac{-\cos\theta}{\sin^2\theta}$$

$$= \frac{1}{\sin\theta}\cdot\frac{-\cos\theta}{\sin\theta}$$

$$\frac{d}{d\theta}(\csc\theta) = -\csc\theta\cot\theta$$

Find the derivative of each of the functions below by applying the quotient rule.

$$f(x) = \frac{x^2 - 2x}{x+2}$$

$$p(\theta) = \frac{\sin\theta}{\theta^2}$$

$h(\theta) = \dfrac{\sin\theta}{1-\cos\theta}$

$f(x) = \dfrac{x \cdot \left(3 - \frac{1}{x}\right) \cdot x}{x \cdot (x+5) \cdot x} = \dfrac{3x-1}{x^2+5x}$

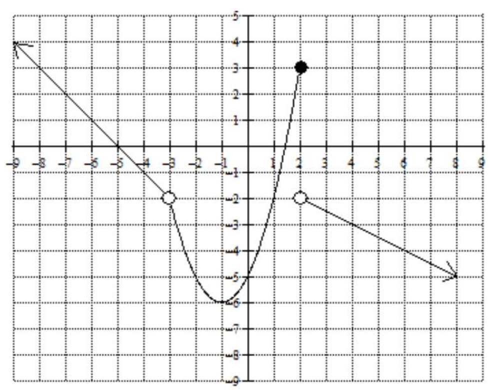
$f'(x) = \dfrac{(3)(x^2+5x) - (3x-1)(2x+5)}{(x^2+5x)^2}$

$= \dfrac{3x^2+15x - [6x^2+15x-2x-5]}{(x^2+5x)^2}$

$= \dfrac{3x^2+\cancel{15x}-6x^2-\cancel{15x}+2x+5}{(x^2+5x)^2}$

$f'(x) = \dfrac{-3x^2+2x+5}{(x^2+5x)^2}$

Below is a graph of a function, $f(x)$ and let $R(x) = \dfrac{\cos x}{f(x)}$. Is the graph of $R(x)$ increasing, decreasing or neither increasing nor decreasing at $x = \pi$? Show your work and justify your reasoning.



$R'(x) = \dfrac{(-\sin x) f(x) - \cos x \cdot f'(x)}{f^2(x)}$

$R'(\pi) = \dfrac{-\sin\pi \cdot f(\pi) - \cos\pi \cdot f'(\pi)}{f^2(\pi)}$

$= \dfrac{-(0)\cdot f(\pi) - (-1)\cdot(-\frac{1}{2})}{(\text{neg})^2}$

$= \dfrac{0 - \frac{1}{2}}{\text{pos}}$

$R'(\pi) = \dfrac{-\frac{1}{2}}{\text{pos}}$

$R'(\pi) = \text{neg}$

$R(x)$ is decreasing at $x = \pi$, b/c $R'(\pi) < 0$.

Let $f(x)$ and $g(x)$ be differentiable functions such that the following values are true.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	2	–1	9	–1
3	–5	–3	–4	6
4	1	7	8	–2

Estimate the value of $g'(2.5)$.

$$g'(2.5) \approx \frac{g(3)-g(2)}{3-2}$$

$$\approx \frac{(-3)-(-1)}{1}$$

$$\approx \frac{-2}{1}$$

$$g'(2.5) \approx -2$$

If $p(x) = \dfrac{g(x)}{f(x)}$, what is the value of $p'(4)$? What does this value say about the graph of $p(x)$ when $x = 4$? Give a reason for your answer.

$$p'(x) = \frac{g'(x)\,f(x) - g(x)\cdot f'(x)}{f^2(x)}$$

$$p'(4) = \frac{g'(4)\cdot f(4) - g(4)\,f'(4)}{f^2(4)}$$

$$= \frac{(-2)(1) - (7)(8)}{(1)^2}$$

$$= \frac{-2-56}{1}$$

$$p'(4) = -58$$

Since $p'(4) < 0$ this would indicate that $p(x)$ is decreasing at $x = 4$.

If $q(x) = 2x^2\left(\dfrac{f(x)}{g(x)}\right)$, what is the value of $q'(2)$?

1st 2nd

$$q'(x) = 4x\cdot\left(\frac{f(x)}{g(x)}\right) + 2x^2\cdot\left(\frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}\right)$$

$$q'(2) = 4(2)\cdot\frac{f(2)}{g(2)} + 2(2)^2\left(\frac{f'(2)\cdot g(2) - f(2)g'(2)}{g^2(2)}\right)$$

$$= 8\cdot\frac{2}{-1} + 8\left(\frac{9\cdot(-1) - (2)(-1)}{(-1)^2}\right)$$

$$= -16 + 8\left(\frac{-9+2}{1}\right)$$

$$= -16 + 8(-7)$$

$$= -16 - 56$$

$$q'(2) = -72$$

Find the equation of the line tangent to the graph of $v(x) = \dfrac{3x}{g(x)}$ when $x = 3$.

PoT $(3,-3)$

$v(3) = \dfrac{3(3)}{g(3)}$

$= \dfrac{9}{-3}$

$v(3) = -3$

SoT

$v'(x) = \dfrac{3 \cdot g(x) - 3x \cdot g'(x)}{g^2(x)}$

$v'(3) = \dfrac{3 \cdot g(3) - 3(3)g'(3)}{g^2(3)}$

$= \dfrac{3(-3) - 9(6)}{(-3)^2}$

$= \dfrac{-9-54}{9}$

$= \dfrac{-63}{9}$

$v'(3) = -7$

Tangent Line

$y+3 = -7(x-3)$