Notes 3.3 – Rules for Differentiation
Finding the Derivative of a Composite Function

Rewrite the function $f(x) = (2x+3)^3$ as a function in polynomial form. Then, find $f'(x)$.

$f(x) = (4x^2 + 12x + 9)(2x+3)$
$= 8x^3 + 12x^2 + 24x^2 + 36x + 18x + 27$
$f(x) = 8x^3 + 36x^2 + 54x + 27$

$f'(x) = 24x^2 + 72x + 54$

Leibniz was the first of the two great calculus developers to use the Chain Rule to differentiate composite functions. Let's write this rule together in the box below.

Chain Rule of Differentiation of Composite Functions

If $f(x) = g(h(x))$, then $\frac{d}{dx} f(x) = \frac{d}{dx} g(h(x)) \cdot \frac{d}{dx} h(x)$

$f' = g'(h(x)) \cdot h'(x)$

To show that this rule works, let's apply this rule to the function $f(x) = (2x+3)^3$ that we rewrote and differentiated as a polynomial-form above.

$f'(x) = 3(2x+3)^2 \cdot (2)$
$f'(x) = 6(2x+3)^2$

$\boxed{\text{Find the slope of the normal line}}$ to the graph of $f(\theta) = \sin^2\theta$ when $\theta = \frac{3\pi}{4}$.

$f(\theta) = [\sin\theta]^2$

SOT

$f'(\theta) = 2[\sin\theta]^1 \cdot \cos\theta$
$f'\left(\frac{3\pi}{4}\right) = 2\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{3\pi}{4}\right)$
$= \cancel{2}\left(\frac{\sqrt{2}}{\cancel{2}}\right)\left(-\frac{\sqrt{2}}{2}\right)$
$f'\left(\frac{3\pi}{4}\right) = -1$

Slope of normal line $= 1$

Find the derivative of each of the following functions by applying the chain rule.

$f(x) = (3x^2+2)^3$

$f'(x) = 3(3x^2+2)^2 \cdot (6x)$

$f'(x) = 18x(3x^2+2)^2$

$g(x) = \sqrt{2x+5} = (2x+5)^{1/2}$

$g'(x) = \frac{1}{2}(2x+5)^{-1/2}(2)$

$g'(x) = \frac{1}{\sqrt{2x+5}}$

$h(x) = \sqrt[3]{(x+2)^2} = (x+2)^{2/3}$

$h' = \frac{2}{3}(x+2)^{-1/3}(1)$

$h' = \frac{2}{3(x+2)^{1/3}}$

$h' = \frac{2}{3\sqrt[3]{x+2}}$

$h(x) = ((x+2)^2)^{1/3}$

$h'(x) = \frac{1}{3}((x+2)^2)^{-2/3} \cdot 2(x+2)^1 \cdot (1)$

$= \frac{1}{3}(x+2)^{-4/3} \cdot 2(x+2)^{3/3}$

$= \frac{2}{3}(x+2)^{-1/3}$

$F(x) = 5\sqrt[3]{x^2+2x} = 5(x^2+2x)^{1/3}$

$F'(x) = \frac{5}{3}(x^2+2x)^{-2/3}(2x+2)$

$= \frac{5(2x+2)}{3\sqrt[3]{(x^2+2x)^2}}$

$F' = \frac{10(x+1)}{3\sqrt[3]{(x^2+2x)^2}}$ OPTIONAL

$G(x) = \cos^2 3x = [\cos(3x)]^2$

$G'(x) = 2\underbrace{[\cos(3x)]^1 \underbrace{(-\sin(3x))\cdot 3}_{\text{Chain}}}_{\text{Chain}}$

$= -6\sin(3x)\cos(3x)$

$h(x) = \sin^2(2x+1) = [\sin(2x+1)]^2$

$h'(x) = 2\cdot\underbrace{\sin(2x+1)\cdot\underbrace{\cos(2x+1)\cdot 2}_{\text{chain}}}$

$= 4\sin(2x+1)\cos(2x+1)$

Now that you know “THE BIG THREE” rules of differentiation—product, quotient, and chain—let’s see how the three can be incorporated with each other. Find the derivative of each of the following functions.

$f(x) = 5x\sqrt{x+3} = 5x\cdot(x+3)^{1/2}$

$$f'(x) = \overbrace{5\cdot(x+3)^{1/2} + 5x\cdot\overbrace{\tfrac{1}{2}(x+3)^{-1/2}(1)}^{\text{Chain}}}^{\text{Product}}$$

$$= 5(x+3)^{-\frac{1}{2}}\left[(x+3)^{1} + \frac{x}{2}\right]$$

$$= \frac{5}{\sqrt{x+3}}\left[\frac{2x}{2} + \frac{x}{2} + \frac{6}{2}\right]$$

$$= \frac{5}{\sqrt{x+3}}\left[\frac{3x+6}{2}\right]$$

$$= \frac{15(x+2)}{2\sqrt{x+3}}$$

$g(x) = \sin\left(\dfrac{2x+1}{x-3}\right)$

$$g'(x) = \overbrace{\cos\left(\frac{2x+1}{x-3}\right)\cdot\overbrace{\left(\frac{2\cdot(x-3)-(2x+1)(1)}{(x-3)^2}\right)}^{\text{Quotient}}}^{\text{Chain Rule}}$$

$$= \cos\left(\frac{2x+1}{x-3}\right)\cdot\left(\frac{\cancel{2x}-6-\cancel{2x}-1}{(x-3)^2}\right)$$

$$g' = \frac{-7}{(x-3)^2}\cos\left(\frac{2x+1}{x-3}\right)$$

$$h(x) = \frac{\sqrt{2x+5}}{x-3} = \frac{(2x+5)^{1/2}}{x-3}$$

$$h'(x) = \frac{\left[(2x+5)^{\frac{1}{2}}\right]' \cdot (x-3) - (2x+5)^{1/2}(x-3)'}{(x-3)^2}$$

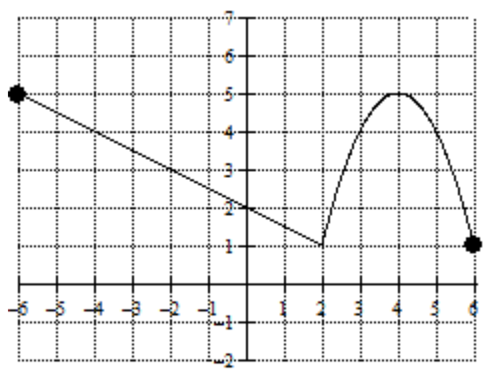
Chain

$$h'(x) = \frac{\frac{1}{2}(2x+5)^{-\frac{1}{2}}(2) \cdot (x-3) - (2x+5)^{\frac{1}{2}}(1)}{(x-3)^2}$$ Quotient Rule

$$= \frac{(2x+5)^{-\frac{1}{2}}\left[1(x-3) - (2x+5)^1\right]}{(x-3)^2}$$

$$h' = \frac{-x-8}{\sqrt{2x+5}\,(x-3)^2}$$

Given the graph of $H(x)$ pictured to the right, find the equation of the tangent line to the graph of $P(x) = \sqrt{H(x)}$ when $x = -4$.



PoT $(-4, 2)$

$$P(-4) = \sqrt{H(-4)}$$
$$= \sqrt{4}$$
$$P(-4) = 2$$

SoT $m = -\frac{1}{8}$

$$P'(x) = \frac{1}{2}(H(x))^{-\frac{1}{2}} \cdot H'(x)$$
$$P'(-4) = \frac{H'(-4)}{2\sqrt{H(-4)}}$$
$$= \frac{-\frac{1}{2}}{2 \cdot 2}$$
$$= \frac{-\frac{1}{2}}{4}$$
$$P'(-4) = -\frac{1}{8}$$

Tangent line

$$y - 2 = -\frac{1}{8}(x+4)$$

Let $f(x)$ and $g(x)$ be differentiable functions such that the following values are true.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	2	–1	0	–1
3	–5	4	–4	6
4	1	7	8	–2

Is the graph of $h(x) = f(g(x))$ increasing, decreasing or at a relative maximum or minimum when $x = 3$? Give a reason for your answer.

$h'(x) = f'(g(x)) \cdot g'(x)$

$h'(3) = f'(g(3)) \cdot g'(3)$

$= f'(4) \cdot 6$

$= 8 \cdot 6$

$h'(3) = 48$

h(x) is increasing at $x=3$ because $h'(3) > 0$

If $p(x) = g(2x)$, what is the value of $p'(1)$?

$p'(x) = g'(2x) \cdot 2$

$p'(1) = g'(2 \cdot 1) \cdot 2$

$= 2 \cdot g'(2)$

$= 2(-1)$

$p'(1) = -2$

If $q(x) = \sqrt{f(x) + g(x)}$, what is the value of $q'(4)$? What does this value say about the graph of $q(x)$ when $x = 4$? Give a reason for your answer.

$q(x) = (f(x)+g(x))^{\frac{1}{2}}$

$q'(x) = \frac{1}{2}(f(x)+g(x))^{-\frac{1}{2}}(f'(x)+g'(x))$

$= \dfrac{f'(x)+g'(x)}{2\sqrt{f(x)+g(x)}}$

$q'(4) = \dfrac{f'(4)+g'(4)}{2\sqrt{f(4)+g(4)}}$

$= \dfrac{8+(-2)}{2\sqrt{1+7}}$

$q'(4) = \dfrac{6}{2\sqrt{8}}$

The value of $q'(4) > 0$, so $q(x)$ is increasing at $x=4$