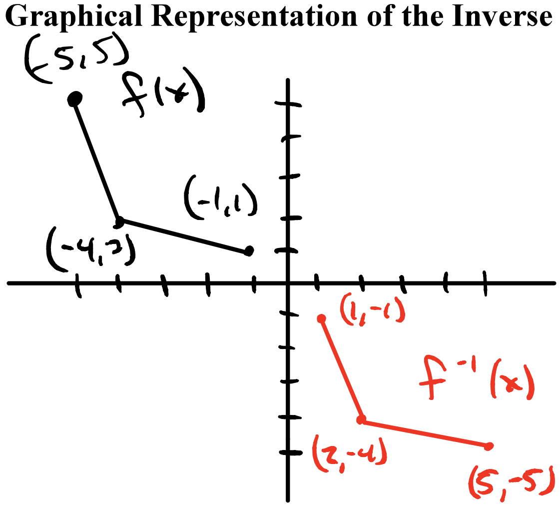
Notes 3.6 – Derivatives of Inverse Functions

From earlier courses, let's take just a moment and remember what inverse functions are. Given a function, $f(x)$, the inverse function, $f^{-1}(x)$, is numerically defined to be the function whose domain is the range of $f(x)$ and whose range is the domain of $f(x)$.

Graphical Representation of the Inverse



Analytical Representation of the Inverse

If $f^{-1}(x)$ is the inverse of $f(x)$, then $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

Consider the two functions, $f(x)$ and $g(x)$, represented numerically below. Answer the questions that follow.

$f(x)$	$f^{-1}(x)$
(-2,3)	(3,-2)
(1,2)	(2,1)

x	$f(x)$	$g(x)$
−2	3	1
1	2	−2

$g(x)$	$g^{-1}(x)$
(-2,1)	(1,-2)
(1,-2)	(-2,1)

Complete the table of values below.

x	$f^{-1}(x)$
3	-2
2	1

Complete the table of values below.

x	$g^{-1}(x)$
1	-2
-2	1

Find the value of $f\left(g^{-1}(-2)\right)$.

$f(g^{-1}(-2)) = f(1) = 2$

Find the value of $f^{-1}\left(f(1)\right)$.

$f^{-1}(f(1)) = f^{-1}(2)$

$= 1$

Find the value of $g^{-1}\left(f^{-1}(2)\right)$.

$= g^{-1}(1)$

$= -2$

Find the value of $f^{-1}\left(g\left(g^{-1}(1)\right)\right)$.

$= f^{-1}(1)$

$=$ DNE

Given below is the relationship that exists when a composite function is formed using a function and its inverse. Use the chain rule to differentiate both sides of the equation and find the formula for the derivative of $f^{-1}(x)$.

Finding a Formula for the Derivative of an Inverse

Differentiate both sides of the equation below.

$$f[f^{-1}(x)] = x$$

$$f'(f^{-1}(x)) \cdot [f^{-1}(x)]' = 1$$

$$f^{-1}(x) = \frac{1}{f'[f^{-1}(x)]}$$

Suppose that $f(x) = 3x + 2$ and $f'(-2) = 3$. What is the value of $[f^{-1}(-4)]'$?

1. $f^{-1}(x)$

$$\boxed{\begin{aligned} x &= 3y + 2 \\ x - 2 &= 3y \\ \tfrac{1}{3}(x-2) &= y \end{aligned}}$$

2. $f^{-1}(x) = \frac{1}{3}(x-2)$

$f^{-1}(-4) = \frac{1}{3}(-4-2)$

$= \frac{1}{3}(-6)$

$f^{-1}(-4) = -2$

3. $[f^{-1}(x)]' = \dfrac{1}{f'[f^{-1}(x)]}$

$[f^{-1}(-4)]' = \dfrac{1}{f'[f^{-1}(-4)]}$

$= \dfrac{1}{f'[-2]}$

$= \dfrac{1}{3}$

Given to the right is a table of values for f, g, f', and g'. Use the values in the table to find each indicated value in the boxes below.

f^{-1}	x	f	g	f'	g'	g^{-1}
(1,-2)	-2	1	2	0	3	(2,-2)
(-4,0)	0	-4	-3	-1	2	(-3,0)
(3,1)	1	3	-2	2	1	(-2,1)
(1,3)	3	1	1	-3	-2	(1,3)

Find the value of $[f^{-1}(3)]'$.

$[f^{-1}(x)]' = \dfrac{1}{f'[f^{-1}(x)]}$

$[f^{-1}(3)]' = \dfrac{1}{f'[f^{-1}(3)]}$

$= \dfrac{1}{f'[1]}$

$[f^{-1}(3)]' = \dfrac{1}{2}$

Find the value of $[g^{-1}(-2)]'$.

$[g^{-1}(x)]' = \dfrac{1}{g'[g^{-1}(x)]}$

$[g^{-1}(-2)]' = \dfrac{1}{g'[g^{-1}(-2)]}$

$= \dfrac{1}{g'[1]}$

$= \dfrac{1}{1}$

$[g^{-1}(-2)]' = 1$

The functions, f and g, are differentiable functions and selected values of them and their first derivatives, f' and g', are shown in the table below. Use the table of values to answer the questions that follow.

f^{-1}	x	f	g	f'	g'	g^{-1}
(1, −2)	−2	1	2	0	3	(2, −2)
(−4, 0)	0	−4	−3	−1	2	(−3, 0)
(3, 1)	1	3	−2	2	1	(−2, 1)
(1, 3)	3	1	1	−3	−2	(1, 3)

Find the value of $[g^{-1}(1)]'$. Then, find the equation of the line tangent to the graph of g^{-1} when $x = 1$.

PoT (1, 3) SoT $m = -\frac{1}{2}$

Table

$$[g^{-1}(x)]' = \frac{1}{g'[g^{-1}(x)]}$$

$$[g^{-1}(1)]' = \frac{1}{g'[g^{-1}(1)]}$$

$$= \frac{1}{g'(3)}$$

$$= \frac{1}{-2}$$

Tangent line

$$y - 3 = -\frac{1}{2}(x - 1)$$

Estimate the value of $f'(2)$. Based on this value, what conclusion can be reached about the graph of f when $x = 2$? Explain your reasoning.

$$f'(2) \approx \frac{f(3) - f(1)}{3 - 1} = \frac{(1) - (3)}{2} = -2/2 = -1$$

$$f'(2) \approx -1$$

The graph is decreasing at $x = 2$ b/c $f'(2) < 0$

Estimate the value of $g'(-1)$. Based on this value, what conclusion can be reached about the graph of g when $x = -1$? Explain your reasoning.

$$g'(-1) \approx \frac{g(-2) - g(0)}{(-2) - (0)} = \frac{(2) - (-3)}{-2} = \frac{5}{-2}$$

The graph is decreasing at $x = -1$ b/c $g'(-1) < 0$