FREE RESPONSE #1

Consider the piece-wise defined function below to answer the questions that follow.

$$f(x)=\begin{cases} ax^2+bx+2, & x\le 2 \\ ax+b, & x>2 \end{cases}$$

a. If $a=-3$ and $b=4$, will $f(x)$ be continuous at $x=2$? Justify your answer.

$f(x)=\begin{cases} -3x^2+4x+2, & x\le 2 \\ -3x+4, & x>2 \end{cases}$

I. $f(2)=-3(2)^2+4(2)+2$
$=-3(4)+8+2$
$=-12+10$
$f(2)=-2$ $\therefore$ $f(2)$ is defined.

II. $\lim_{x\to 2^-} f(x)=-2$
$\lim_{x\to 2} f(x)=-3(2)+4$
$=-6+4$
$=-2$
$\therefore \lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=-2$

III $f(2)=\lim_{x\to 2} f(x)=-2$

$\therefore$ $f(x)$ is continuous at $x=2$

b. If $a=-3$ and $b=4$, will $f(x)$ be differentiable at $x=2$? Justify your answer.

$f'(x)=\begin{cases} -6x+4, & x\le 2 \\ -3, & x>2 \end{cases}$

1. $f(x)$ is continuous at $x=2$.

2. $\lim_{x\to 2^-} f'(x)=-6(2)+4$
$=-12+4$
$=-8$

$\lim_{x\to 2^+} f'(x)=-3$

$\lim_{x\to 2^-} f(x)\neq\lim_{x\to 2^+} f(x)$ $\therefore$ $f(x)$ is not differentiable at $x=2$

c. For what value(s) of a and b will $f(x)$ be both continuous and differentiable at $x=2$? Show your work.

1 $f(x)=\begin{cases} ax^2+bx+2, & x\le 2 \\ ax+b, & x>2 \end{cases}$

$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)$

$a(2)^2+b(2)+2=a(2)+b$

$4a+2b+2=2a+b$

$2a+b+2=0$

$2a+b=-2$ $\xrightarrow{\text{Mult }-1}$ 3 $-2a-b=+2$

$3a+b=0$

$a=2$

2 $f'(x)=\begin{cases} 2ax+b, & x\le 2 \\ a, & x>2 \end{cases}$

$\lim_{x\to 2^-} f'(x)=\lim_{x\to 2^+} f'(x)$

$2a(2)+b=a$

$4a+b=a$

$3a+b=0$

4 $3(2)+b=0$

$b=-6$

$\therefore$ For $f(x)$ to be both continuous and differentiable at $x=2$,

$a=2$ and $b=-6$

CALC

FREE RESPONSE #2

A rodeo performer spins a lasso in a circle perpendicular to the ground. The height from the ground of the knot, measured in units of feet, in the lasso is modeled by the function

$$H(t) = -3\cos\left(\frac{5\pi}{3}t\right) + 5,$$

where t is the time measured in seconds after the lasso begins to spin.

a. Find the value of $H(0.75)$. Using correct units, explain what this value represents in the context of this problem.

$H(0.75) = 7.121$ feet
Seconds

At 0.75 seconds the height of the lasso above the ground is 7.121 feet.

b. Find the value of $H'(0.75)$. Using correct units, explain what this value represents in the context of this problem.

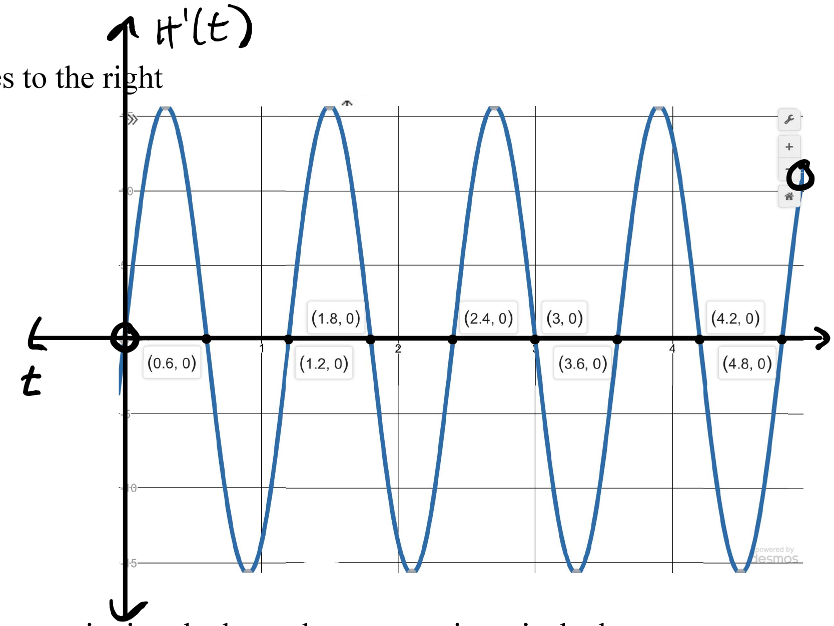
$H'(0.75) \approx -11.107$

At 0.75 seconds the height of the lasso above the ground is decreasing by 11.107 feet per second.

c. Find $H'(t)$ and sketch its graph on the axes to the right for the interval $0 < t < 5$ seconds.

$$H'(t) = -3\underbrace{\left[-\sin\left(\frac{5\pi}{3}t\right)\right]\cdot\frac{5\pi}{3}}_{\text{Chain}}$$

$$H'(t) = 5\pi\sin\left(\frac{5\pi}{3}t\right)$$



d. During the first five seconds of the performer spinning the lasso, how many times is the lasso at its maximum height? Give a reason for your answer based on the graph of $H'(t)$.

The lasso reaches its maximum height 4 times because there are four times during the first 5 seconds where the graph of H'(t) changes from positive to negative.

e. What is the height of the lasso the first time it is at its minimum height on the interval $0 < t < 5$ seconds? Justify your answer and show your work.

At t = 1.2 is the first time H'(t) changes from negative, thus H(1.2) = 2 feet is a minimum.