9.3 Average Value of a Function

Calculus Name: ______________________________ **Practice**

Find the average value of each function on the given interval.

1. $f(x) = x^2$ on $[2, 4]$

$$Av = \frac{1}{4-2}\int_2^4 x^2\,dx = \frac{1}{2}\cdot\frac{1}{3}x^3\Big|_2^4$$
$$= \left[\frac{1}{6}(4)^3\right] - \left[\frac{1}{6}(2)^3\right]$$
$$= \frac{64}{6} - \frac{8}{6}$$
$$= \frac{56}{6}$$
$$Av = \frac{28}{3}$$

2. $f(x) = x^2 - 2x$ on $[0, 3]$

$$Av = \frac{1}{3-0}\int_0^3 (x^2 - 2x)\,dx = \frac{1}{3}\left(\frac{1}{3}x^3 - x^2\right)\Big|_0^3$$
$$= \left[\frac{1}{3}\left(\frac{1}{3}(3)^3 - (3)^2\right)\right] - \left[\frac{1}{3}\left(\frac{1}{3}(0)^3 - (0)^2\right)\right]$$
$$= \left[\frac{1}{3}\left(\frac{27}{3} - 9\right)\right] - \left[\frac{1}{3}\left(\frac{0}{3} - 0\right)\right]$$
$$= \left[\frac{1}{3}(9-9)\right] - \left[\frac{1}{3}(0)\right]$$
$$= \left[\frac{1}{3}(0)\right] - 0$$
$$= 0 - 0$$
$$Av = 0$$

3. $f(x) = \sin x$ on $[0, \pi]$

$$Av = \frac{1}{\pi - 0}\int_0^\pi \sin x\,dx$$
$$= \frac{1}{\pi}(-\cos x)\Big|_0^\pi$$
$$= \left[-\frac{1}{\pi}\cos\pi\right] - \left[-\frac{1}{\pi}\cos 0\right]$$
$$= \left[-\frac{1}{\pi}\cdot(-1)\right] - \left[-\frac{1}{\pi}(1)\right]$$
$$= \left[\frac{1}{\pi}\right] + \left[\frac{1}{\pi}\right]$$
$$Av = \frac{2}{\pi}$$

4. $f(x) = \sqrt{x}$ on $[0, 16]$

$$Av = \frac{1}{16-0}\int_0^{16} x^{\frac{1}{2}}\,dx = \frac{1}{\cancel{16}_{8}}\left(\frac{\cancel{2}}{3}\right)x^{\frac{3}{2}}\Big|_0^{16}$$
$$= \left[\frac{1}{24}(\sqrt{16})^3\right] - \left[\frac{1}{24}(\sqrt{0})^3\right]$$
$$= \left[\frac{1}{24}(4)^3\right] - \left[\frac{1}{24}(0)\right]$$
$$= \left[\frac{1}{24}(64)\right] - [0]$$
$$Av = \frac{8}{3}$$

5. $f(x) = \frac{1}{x^2}$ on $[-4, -2]$

$$Av = \frac{1}{-2-(-4)}\int_{-4}^{-2} x^{-2}\,dx = \frac{1}{2}(-1)x^{-1}\Big|_{-4}^{-2}$$
$$= \frac{-1}{2x}\Big|_{-4}^{-2}$$
$$= \left[\frac{-1}{2(-2)}\right] - \left[\frac{-1}{2(-4)}\right]$$
$$= \left[\frac{1}{4}\right] - \left[\frac{1}{8}\right]$$
$$= \frac{2}{8} - \frac{1}{8}$$
$$Av = \frac{1}{8}$$

6. $f(x) = 2e^x$ on $[-3, 1]$

$$Av = \frac{1}{1-(-3)}\int_{-3}^{1} 2e^x\,dx = \frac{1}{4}2e^x\Big|_{-3}^{1}$$
$$= \left[\frac{1}{2}e^1\right] - \left[\frac{1}{2}e^{-3}\right]$$
$$Av = \frac{e}{2} - \frac{1}{2e^3}$$
or
$$= \frac{e^4}{2e^3} - \frac{1}{2e^3}$$
$$Av = \frac{e^4 - 1}{2e^3}$$

Find the value(s) c that satisfy the MVT for *integrals*.

7. $f(x) = 2x - 2$ on $[1, 4]$

$$Av = \frac{1}{4-1}\int_1^4 (2x-2)\,dx = \frac{1}{3}(x^2 - 2x)\Big|_1^4$$
$$= \left[\frac{1}{3}(4^2 - 2(4))\right] - \left[\frac{1}{3}((1)^2 - 2(1))\right]$$
$$= \left[\frac{1}{3}(16-8)\right] - \left[\frac{1}{3}(1-2)\right]$$
$$= \left[\frac{8}{3}\right] - \left[-\frac{1}{3}\right]$$
$$= \frac{9}{3}$$
$$Av = 3$$

MVT for $\int$

$$f(x) = 2x - 2$$
$$3 = 2x - 2$$
$$5 = 2x$$
$$\frac{5}{2} = x$$

8. $f(x) = -\frac{x^2}{2}$ on $[0, 3]$

$$Av = \frac{1}{3-0}\int_0^3 -\frac{1}{2}x^2\,dx = \frac{1}{3}\left(-\frac{1}{2}\right)\frac{1}{3}x^3\Big|_0^3$$
$$= \left[-\frac{1}{18}(3)^3\right] - \left[-\frac{1}{18}(0)^3\right]$$
$$= \left[\frac{-27}{18}\right] - [0]$$
$$Av = \frac{-3}{2}$$

MVT $\int$

$$f(x) = -\frac{1}{2}x^2$$
$$-\frac{3}{2} = -\frac{1}{2}x^2$$
$$3 = x^2$$
$$\pm\sqrt{3} = x$$
$$-\sqrt{3} \notin [0,3] \text{ so } x = \sqrt{3}$$

9. $f(x) = 2x^2 + 16x + 28$ on $[-5, -2]$

$$Av = \frac{1}{-2-(-5)}\int_{-5}^{-2}(2x^2 + 16x + 28)\,dx = \frac{1}{3}\left(\frac{2}{3}x^3 + 8x^2 + 28x\right)\Big|_{-5}^{-2}$$
$$= \left[\frac{1}{3}\left(\frac{2}{3}(-2)^3 + 8(-2)^2 + 28(-2)\right)\right] - \left[\frac{1}{3}\left(\frac{2}{3}(-5)^3 + 8(-5)^2 + 28(-5)\right)\right]$$
$$= \left[\frac{1}{3}\left(\frac{2}{3}(-8) + 8(4) - 56\right)\right] - \left[\frac{1}{3}\left(\frac{2}{3}(-125) + 8(25) - 140\right)\right]$$
$$= \left[\frac{1}{3}\left(\frac{-16}{3} + 32 - 56\right)\right] - \left[\frac{1}{3}\left(\frac{-250}{3} + 200 - 140\right)\right]$$
$$= \left[\frac{1}{3}\left(\frac{-16}{3} - 24\right)\right] - \left[\frac{1}{3}\left(\frac{-250}{3} + 60\right)\right]$$
$$= \frac{-16}{9} - 8 + \frac{250}{9} - 20$$
$$= \frac{234}{9} - 28$$
$$= 26 - 28$$
$$Av = -2$$

MVT $\int$

$$f(x) = 2x^2 + 16x + 28$$
$$-2 = 2x^2 + 16x + 28$$
$$0 = 2x^2 + 16x + 30$$
$$0 = x^2 + 8x + 15$$
$$0 = (x+5)(x+3)$$
$$0 = x+5 \quad | \quad 0 = x+3$$
$$-5 = x \quad | \quad -3 = x$$
$$x = -5, -3$$

Find the average rate of change on the given interval.

10. $f(x) = -(2x - 6)^{\frac{2}{3}}$ on $[1, 3]$

$$ARC = \frac{\Delta y}{\Delta x} = \frac{f(3) - f(1)}{3 - 1}$$
$$= \frac{0 - (-2\sqrt[3]{2})}{2}$$
$$= \frac{2\sqrt[3]{2}}{2}$$
$$ARC = \sqrt[3]{2}$$

$$f(3) = -\left(\sqrt[3]{2(3) - 6}\right)^2$$
$$= -\left(\sqrt[3]{0}\right)^2$$
$$f(3) = 0$$

$$f(1) = -\left(\sqrt[3]{2(1) - 6}\right)^2$$
$$= -\left(\sqrt[3]{-4}\right)^2$$
$$= -\sqrt[3]{16}$$
$$f(1) = -2\sqrt[3]{2}$$

11. $y = x^3 - 2x^2 + 2$ on $[-1, 1]$

$$ARC = \frac{f(1) - f(-1)}{1 - (-1)}$$
$$= \frac{1 - (-1)}{2}$$
$$= \frac{2}{2}$$
$$ARC = 1$$

$$f(1) = (1)^3 - 2(1)^2 + 2$$
$$= 1 - 2(1) + 2$$
$$= 1 - 2 + 2$$
$$f(1) = 1$$

$$f(-1) = (-1)^3 - 2(-1)^2 + 2$$
$$= -1 - 2(1) + 2$$
$$= -1 - 2 + 2$$
$$f(-1) = -1$$

12. $y = \ln\sqrt{x}$ on $[1, e]$

$$ARC = \frac{f(e) - f(1)}{e - 1}$$
$$= \frac{\ln\sqrt{e} - \ln\sqrt{1}}{e - 1}$$
$$= \frac{\frac{1}{2}\ln e - \ln 1}{e - 1}$$
$$= \frac{\frac{1}{2}(1) - 0}{e - 1}$$
$$= \frac{\frac{1}{2}}{e - 1} \cdot \frac{2}{2}$$
$$ARC = \frac{1}{2e - 2}$$

For 13-14, find the value(s) c that satisfy the MVT for *derivatives*.

13. $y = x^2 - 4x + 3$ on $[0, 4]$

$ARC = y'(c)$

$\frac{y(4)-y(0)}{4-0} = 2c-4$

$\frac{[(4)^2-4(4)+3]-[(0)^2-4(0)+3]}{4} = 2c-4$

$\frac{[16-16+3]-[3]}{4} = 2c-4$

$\frac{3-3}{4} = 2c-4$

$\frac{0}{4} = 2c-4$

$0 = 2c-4$

$4 = 2c$

$2 = c$

14. $y = \sqrt{9-3x}$ on $[-2, 3]$

CHAIN RULE

$y' = \frac{1}{2}(9-3x)^{-\frac{1}{2}}(-3)$

$y' = \frac{-3}{2\sqrt{9-3x}}$

$ARC = \frac{y(3)-y(-2)}{3-(-2)}$

$= \frac{\sqrt{9-3(3)} - \sqrt{9-3(-2)}}{5}$

$= \frac{\sqrt{9-9} - \sqrt{9+6}}{5}$

$= \frac{0-\sqrt{15}}{5}$

$ARC = \frac{-\sqrt{15}}{5}$

$ARC = y'(c)$

$-\frac{\sqrt{15}}{5} = \frac{-3}{2\sqrt{9-3c}}$

$-2\sqrt{15}\sqrt{9-3c} = -15$

$-\sqrt{9-3c} = \frac{-15}{2\sqrt{15}}$

$9-3c = \frac{225}{4 \cdot 15}$

$9-3c = \frac{15}{4}$

$-3c = \frac{15}{4} - \frac{36}{4}$

$(-\frac{1}{3})-3c = \frac{-21}{4}(-\frac{1}{3})$

$c = \frac{7}{4}$

15. The temperature (in °F) t hours after 9 AM is approximated by the function $T(t) = 50 + 14\sin\frac{\pi t}{12}$. Find the average temperature during the time period 9 AM to 9 PM.

t = hours after 9 AM
$[0, 12]$
T = temp in °F

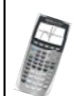
$AT = \frac{1}{12-0}\int_0^{12}\left(50+14\sin\frac{\pi t}{12}\right)dt \approx \frac{1}{12}(706.95212)$

$\approx 58.913°F$

The average temperature from 9 AM to 9 PM was about 58.913°F.

16. The depth of water in Mr. Brust's hot tub can be represented by the formula $h(t) = -\cos(t) + 2$, where t is the time in minutes since he begins pouring in water and $h(t)$ is measured in feet. What is the average depth of the water during the first three minutes? Set up the expression and use a calculator to help solve.

t = time in minutes
h = height in feet



$AD = \frac{1}{3-0}\int_0^3 [-\cos(t)+2]\,dt$

$\approx \frac{1}{3}\{5.85888\}$

$AD \approx 1.953$ feet

The average depth of the hot tub is 1.953 feet

17. Find the number(s) b such that the average value of $y = 2 + 7x - x^3$ on the interval $[0, b]$ is equal 2.

$Av = \frac{1}{b-0}\int_0^b (2+7x-x^3)dx = \frac{1}{b}\left(2x+\frac{7}{2}x^2-\frac{1}{4}x^4\right)\Big|_0^b$

$= \left[\frac{1}{b}\left(2b+\frac{7}{2}b^2-\frac{1}{4}b^4\right)\right] - \left[\frac{1}{b}\left(2(0)+\frac{7}{2}(0)^2-\frac{1}{4}(0)^4\right)\right]$

$= \left[2+\frac{7}{2}b-\frac{1}{4}b^3\right] - \left[\frac{1}{b}(0)\right]$

$Av = 2+\frac{7}{2}b-\frac{1}{4}b^3$

$2 = 2+\frac{7}{2}b-\frac{1}{4}b^3$

$0 = \frac{7}{2}b-\frac{1}{4}b^3$

$0 = b\left(\frac{7}{2}-\frac{1}{4}b^2\right)$

$0 = b$

$0 = \frac{7}{2}-\frac{1}{4}b^2$

$-\frac{7}{2} = -\frac{1}{4}b^2$

$14 = b^2$

$\pm\sqrt{14} = b$

$\sqrt{14} = b$

$b \neq -\sqrt{14}$ because $-\sqrt{14}$ is before 0.

$b \neq 0$ because $[0,0]$ is not an interval

18. Find the number(s) b such that the average value of $y = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal 3. *Hint*: quadratic formula needed!

$Av = \frac{1}{b-0}\int_0^b (2+6x-3x^2)dx = \frac{1}{b}\left(2x+3x^2-x^3\right)\Big|_0^b$

$= \left[\frac{1}{b}\left(2b+3b^2-b^3\right)\right] - \left[\frac{1}{b}\left(2(0)+3(0)^2-(0)^3\right)\right] = \left[2+3b-b^2\right] - \left[\frac{1}{b}(0)\right]$

$Av = 2+3b-b^2$

$3 = -b^2+3b+2$

$b^2-3b+1 = 0$

$b = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$b = \frac{-(-3)\pm\sqrt{(-3)^2-4(1)(1)}}{2(1)}$

$b = \frac{3\pm\sqrt{9-4}}{2}$

$b = \frac{3\pm\sqrt{5}}{2}$

19. 2004 A Q1 c-d

Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function F defined by

$$F(t) = 82 + 4\sin\left(\frac{t}{2}\right) \text{ for } 0 \le t \le 30,$$

where $F(t)$ is measured in cars per minute and t is measured in minutes.

(c) What is the average value of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

$Av = \frac{1}{15-10}\int_{10}^{15}\left(82+4\sin\frac{t}{2}\right)dt \approx \frac{1}{5}(409.496)$

≈ 81.899 cars/min

(d) What is the average rate of change of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

$ARC = \frac{f(15)-f(10)}{15-0} = \frac{85.752-78.164}{5} = \frac{7.588}{5} = 1.518$ cars/min²

20. 2005 A Q3 b-d

$\Delta x_1 = 1 \quad \Delta x_2 = 4 \quad \Delta x_3 = 1 \quad \Delta x_4 = 2$

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100 b_1	93 b_2	70 b_3	62 b_4	55 b_5

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

$$AT = \frac{1}{8}\int_0^8 T(x)\,dx \approx \frac{1}{8}\left[\Delta x_1 \cdot \tfrac{1}{2}(b_1+b_2) + \Delta x_2 \tfrac{1}{2}(b_2+b_3) + \Delta x_3 \tfrac{1}{2}(b_3+b_4) + \Delta x_4 \tfrac{1}{2}(b_4+b_5)\right]$$
$$\approx \frac{1}{8}\left[1\cdot\tfrac{1}{2}(193) + 4\cdot\tfrac{1}{2}(163) + 1\cdot\tfrac{1}{2}(132) + 2\cdot\tfrac{1}{2}(117)\right]$$
$$\approx 75.688°C$$

(c) Find $\int_0^8 T'(x)\,dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

$$\int_0^8 T'(x)dx = T(x)\Big|_0^8 = T(8) - T(0) = 55 - 100 = -45°C$$

The temperature drops 45°C from the heated end to the other end.

(d) Are the data in the table consistent with the assertion that $T''(x) > 0$ for every x in the interval $0 < x < 8$? Explain your answer.

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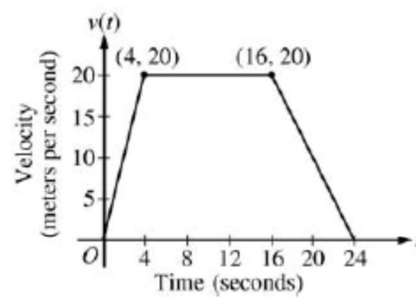
21. 2005 A Q5 d

A car is traveling on a straight road. For $0 \le t \le 24$ seconds, the car's velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph to the right.



(d) Find the average rate of change of v over the interval $8 \le t \le 20$. Does the Mean Value Theorem guarantee a value of c, for $8 < c < 20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?

9.3 Average Value of a Function

Test Prep

1. The average value of $f(x) = x^3$ over the interval $a \le x \le b$ is

$$Av = \frac{1}{b-a}\int_a^b x^3 dx = \frac{1}{b-a}\left(\frac{1}{4}\right)x^4\Big|_a^b = \left[\frac{1}{b-a}\left(\frac{1}{4}\right)b^4\right] - \left[\frac{1}{b-a}\left(\frac{1}{4}\right)a^4\right]$$

(A) $3b + 3a$

(B) $b^2 + ab + a^2$

(C) $\frac{b^3+a^3}{2}$

(D) $\frac{b^3-a^3}{2}$

(E) $\frac{b^4-a^4}{4(b-a)}$

2. The average value of the function $f(x) = \ln^2 x$ on the interval $[2, 4]$ is

(A) -1.204 (B) 1.204 (C) 2.159 (D) 2.408 (E) 8.636

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3. The function f is continuous on the closed interval $[1, 3]$ and has the values given in the table. The equation $f(x) = \frac{5}{4}$ must have at least two solutions in the interval $[1, 3]$ if $k =$

x	1	2	3
$f(x)$	2	k	4

(A) $\frac{1}{4}$ (B) $\frac{3}{2}$ (C) 2 (D) $\frac{9}{4}$ (E) 3

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4. A particle moves along the x-axis so that its position at time t is given by $x(t) = t^2 - 7t + 12$. For what value of t is the velocity of the particle zero?

$v = 2t - 7$
$0 = 2t - 7$
$7 = 2t$
$3.5 = t$

(A) 2.5 (B) 3 (C) 3.5 (D) 4 (E) 4.5

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5. The function g is given by $g(x) = \frac{3x^2}{e^{3x}}$. On which of the following intervals is g increasing?

$g' = \frac{6x \cdot e^{3x} - 3x^2 \cdot 3e^{3x}}{(e^{3x})^2}$ $0 = \frac{3x \cdot e^{3x}(2 - 3x)}{e^{6x}}$

(A) $(-\infty, 0)$ (B) $\left(-\infty, \frac{2}{3}\right)$ (C) $\left(0, \frac{2}{3}\right)$ (D) $(0, \infty)$ (E) $\left(\frac{2}{3}, \infty\right)$

$(-)(+)(+)$ | $(+)(+)(+)$ | $(+)(+)(-)$

$-$ $+$ $-$

0 $2/3$