Topic 1.8 – Determining Limits Using the Squeeze Theorem

1. If $4x-9 \le f(x) \le x^2-4x+7$ for $x \ge 0$, find $\lim_{x\to 4} f(x)$.

- $\lim_{x\to 4}(4x-9) = 4\cdot 4-9 = 16-9 = 7$
- $\lim_{x\to 4}(x^2-4x+7) = 4^2-4(4)+7 = 7$

$\therefore \lim_{x\to 4} f(x) = 7$

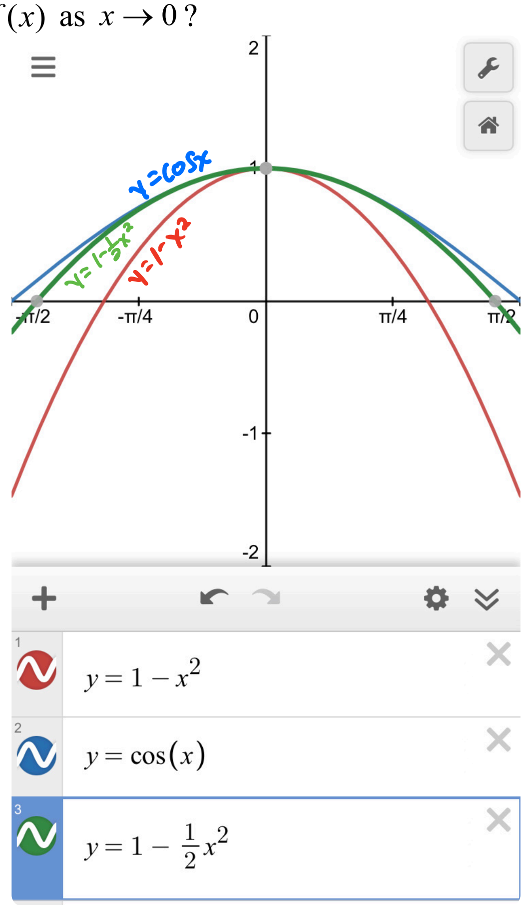
2. If $2x \le g(x) \le x^4-x^2+2$ for all x, find $\lim_{x\to 1} g(x)$.

- $\lim_{x\to 1}(2x) = 2\cdot 1 = 2$
- $\lim_{x\to 1}(x^4-x^2+2) = 1^4-1^2+2 = 2$

$\therefore \lim_{x\to 1} g(x) = 2$

3. Sketch the graphs of the curves $y=1-x^2$, $y=\cos x$, and any arbitrary function $y=f(x)$, where f is a function that satisfies the inequalities $1-x^2 \le f(x) \le \cos x$ for all x in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. What can you say about the limit of $f(x)$ as $x\to 0$?

- $f(x) = 1-\frac{2}{3}x^2$
- $\lim_{x\to 0} f(x) = 1$



Any f(x) of the form $1-ax^2$, where $\frac{1}{2} \le a < 1$, will work

Find the value of each limit. For a limit that does not exist, state why.

4. $\lim_{\theta \to \frac{\pi}{2}} \frac{\cos^2\theta}{1-\sin\theta}$ ② produces indeterminant $\frac{0}{0}$

If $[\cos^2\theta]' = -2\cos\theta\sin\theta$ and $[1-\sin\theta]' = -\cos\theta$

①
$\lim_{\theta\to\frac{\pi}{2}} \cos^2\theta = \left(\cos\frac{\pi}{2}\right)^2 = 0$

$\lim_{\theta\to\frac{\pi}{2}} (1-\sin\theta) = 1-\sin\frac{\pi}{2} = 1-1 = 0$

③ L'Hospital's

$\lim_{\theta\to\frac{\pi}{2}} \frac{-2\cancel{\cos\theta}\sin\theta}{-\cancel{\cos\theta}} = \lim_{\theta\to\frac{\pi}{2}} 2\sin\theta$

$= 2\cdot\sin\left(\frac{\pi}{2}\right)$

$= 2\cdot 1$

$= 2$

5. $\lim_{x\to 0} \frac{x+\sin x}{x}$ ② produces indeterminant $\frac{0}{0}$

If $[x+\sin x]' = 1+\cos x$ and $x' = 1$

①
$\lim_{x\to 0} (x+\sin x) = 0 + \sin 0 = 0$

$\lim_{x\to 0} x = 0$

③ L'Hospital's

$\lim_{x\to 0} \frac{1+\cos x}{1} = 1+\cos(0) = 1+1 = 2$

6. $\lim_{x\to 3} \begin{cases} 2x^2-3x, & x<3 \quad x\to 3^- \\ 8-\cos\left(\frac{\pi x}{3}\right), & x>3 \quad x\to 3^+ \end{cases}$

- $\lim_{x\to 3^-} (2x^2-3x) = 2(3)^2 - 3(3)$
  $= 18-9$
  $= 9$

- $\lim_{x\to 3} \left[8-\cos\left(\frac{\pi x}{3}\right)\right] = 8-\cos\left(\frac{\pi 3}{3}\right)$
  $= 8-\cos\pi$
  $= 8-(-1)$
  $= 9$

$\therefore$ The limit as $x\to 3$ is 9

7. $\lim_{\theta\to 0} \frac{2\sin 3\theta}{\theta}$ ② produces indeterminant $\frac{0}{0}$

If $[2\sin(3\theta)]' = 6\cos(3\theta)$ and $\theta' = 1$

①
- $\lim_{\theta\to 0} 2\sin(3\theta) = 2\sin(3\cdot 0) = 2\sin 0 = 2\cdot 0 = 0$
- $\lim_{\theta\to 0} \theta = 0$

③ L'Hospital's

$\lim_{\theta\to 0} \frac{6\cos(3\theta)}{1} = 6\cos(3\cdot 0)$

$= 6\cos(0)$

$= 6\cdot 1$

$= 6$