Extra Practice – Implicit Differentiation and Related Rates

1. Sand is deposited into a pile with a circular base. The volume V of the pile is given by $V = \frac{r^3}{3}$, where r is the radius of the base, in feet. The circumference of the base is increasing at a constant rate of 5π feet per hour. When the circumference of the base is 8π feet, what is the rate of change of the volume of the pile, in cubic feet per hour?

(A) $\frac{8}{\pi}$ (B) 16 [✓ (C) 40] (D) 40π (E) 80π

$\frac{dC}{dt} = 5\pi$ ft/hr

$$\boxed{\begin{gathered} C = 2\pi r \\ 8\pi = 2\pi r \\ 4 = r \end{gathered}}$$

Find $\frac{dV}{dt}\Big|_{C=8\pi}$

$r=4$

$\frac{dC}{dt} = 2\pi \cdot \frac{dr}{dt}$

$5\pi = 2\pi \cdot \frac{dr}{dt}$

$5/2 = \frac{dr}{dt}$

$V = \frac{1}{3}r^3$

$\frac{dV}{dt} = r^2 \cdot \frac{dr}{dt}$

$\frac{dV}{dt} = (4)^2 \cdot \frac{5}{2}$

$= 16 \cdot \frac{5}{2}$

$\frac{dV}{dt} = 40$

2. If $e^{xy} - y^2 = e - 4$, then at $x = \frac{1}{2}$ and $y = 2$, $\frac{dy}{dx} =$

(A) $\frac{e}{4}$ (B) $\frac{e}{2}$ [✓ (C) $\frac{4e}{8-e}$] (D) $\frac{4e}{4-e}$ (E) $\frac{8-4e}{e}$

$\frac{d}{dx}e^{xy} - \frac{d}{dx}y^2 = \frac{d}{dx}(e-4)$

$e^{xy} \cdot \frac{d}{dx}[x \cdot y] - 2y \cdot \frac{dy}{dx} = 0$

$e^{xy} \cdot \left[1 \cdot y + x \cdot 1 \frac{dy}{dx}\right] - 2y\frac{dy}{dx} = 0$

$y \cdot e^{xy} + xe^{xy}\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx}\left(xe^{xy} - 2y\right) = -ye^{xy}$

$\frac{dy}{dx} = \frac{-ye^{xy}}{xe^{xy} - 2y}$

$\frac{dy}{dx}\Big|_{(\frac{1}{2},2)} = \frac{-2e^1}{\frac{1}{2}e^1 - 2\cdot 2} = \frac{-4e}{e-8}$

3. If $y^3 + y = x^2$, then $\frac{dy}{dx} =$

(A) 0 (B) $\frac{x}{2}$ (C) $\frac{2x}{3y^2}$ (D) $2x - 3y^2$ [✓ (E) $\frac{2x}{1+3y^2}$]

$3y^2 \cdot \frac{dy}{dx} + \frac{dy}{dx} = 2x$

$\frac{dy}{dx}\left(3y^2 + 1\right) = 2x$

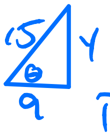
$\frac{dy}{dx} = \frac{2x}{3y^2+1}$

4. The top of a 15-foot-long ladder rests against a vertical wall with the bottom of the ladder on level ground, as shown below. The ladder is sliding down the wall at a constant rate of 2 feet per second. At what rate, in radians per second, is the acute angle between the bottom of the ladder and the ground changing at the instant the bottom of the ladder is 9 feet from the base of the wall?

[✓ (A) $-\frac{2}{9}$] (B) $-\frac{1}{6}$ (C) $-\frac{2}{25}$ (D) $\frac{2}{25}$ (E) $\frac{1}{9}$

$\left.\frac{dy}{dt}\right| = -2$ ft/sec

Find $\left.\frac{d\theta}{dt}\right|_{x=9}$
$y=12$



PT. 3-4-5

$\cos\theta = \frac{9}{15}$

$\sin\theta = \frac{y}{15}$

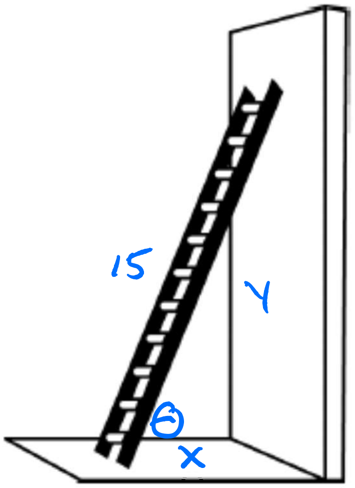
$\frac{d}{dt}\sin\theta = \frac{d}{dt}\frac{y}{15}$

$\cos\theta \cdot \frac{d\theta}{dt} = \frac{1}{15}\cdot\frac{dy}{dt}$

$\frac{9}{15}\cdot\frac{d\theta}{dt} = \frac{1}{15}\cdot -2$

$\frac{d\theta}{dt} = \cancel{\frac{1}{15}}(-2)\frac{\cancel{15}}{9}$

$\frac{d\theta}{dt} = \frac{-2}{9}$



5. Which of the following is an equation of the line tangent to the graph of $x^2 - 3xy = 10$ at the point $(1, -3)$?

(A) $y + 3 = -11(x - 1)$

(B) $y + 3 = -\frac{7}{3}(x - 1)$

(C) $y + 3 = \frac{1}{3}(x - 1)$

(D) $y + 3 = \frac{7}{3}(x - 1)$

[✓ (E) $y + 3 = \frac{11}{3}(x - 1)$]

$2x - \left(3\cdot y - 3x\cdot\frac{dy}{dx}\right) = 0$

$2x - 3y - 3x\frac{dy}{dx} = 0$

$2(1) - 3(-3) - 3(1)\cdot\frac{dy}{dx} = 0$

$2 + 9 - 3\cdot\frac{dy}{dx} = 0$

$-3\frac{dy}{dx} = -11$

$\frac{dy}{dx} = 11/3$

6. If $\ln(2x + y) = x + 1$, then $\frac{dy}{dx} =$

(A) -2 [✓ (B) $2x + y - 2$] (C) $2x + y$ (D) $4x + 2y - 2$ (E) $y - \frac{y}{x}$

$\frac{2 + \frac{dy}{dx}}{2x+y} = 1$

$2 + \frac{dy}{dx} = 2x+y$

$\frac{dy}{dx} = 2x+y-2$

7. The volume of a sphere is decreasing at a constant rate of 3 cubic centimeters per second. At the instant when the radius of the sphere is decreasing at a rate of 0.25 centimeter per second, what is the radius of the sphere?

(The volume V of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.)

(A) 0.141 cm (B) 0.244 cm (C) 0.250 cm (D) 0.489 cm [✓ (E) 0.977 cm]

$$\frac{dV}{dt} = -3 \text{ cm}^3/\text{sec}$$

$$\left.\frac{dr}{dt}\right|_r = -\frac{1}{4} \text{cm}^2/\text{sec}$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$-3 = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$-3 = 4\pi \cdot r^2 \cdot -\frac{1}{4}$$

$$-3 = -\pi r^2$$

$$r^2 = \frac{3}{\pi}$$

$$r = \sqrt{\frac{3}{\pi}}$$

8. If $(x + 2y) \cdot \frac{dy}{dx} = 2x - y$, what is the value of $\frac{d^2y}{dx^2}$ at the point $(3, 0)$?

[✓ (A) $-\frac{10}{3}$] (B) 0 (C) 2 (D) $\frac{10}{3}$ (E) Undefined

$$\frac{dy}{dx} = \frac{2x-y}{x+2y} \Longrightarrow \left.\frac{dy}{dx}\right|_{(3,0)} = \frac{2(3)-0}{3+2(0)} = \frac{6}{3} = 2$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dx}(2x-y)(x+2y) - (2x-y)\frac{d}{dx}(x+2y)}{(x+2y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\left(2-\frac{dy}{dx}\right)(x+2y) - (2x-y)\cdot\left(1+2\frac{dy}{dx}\right)}{(x+2y)^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(3,0)} = \frac{(2-2)(3+2(0)) - (2\cdot 3-0)(1+2(2))}{(3+2(0))^2} = \frac{0-6\cdot 5}{3^2} = \frac{-30}{9} = \frac{-10}{3}$$

9. A person whose height is 6 feet is walking away from the base of a streetlight along a straight path at a rate of 4 feet per second. If the height of the streetlight is 15 feet, what is the rate at which the person's shadow is lengthening?

(A) 1.5 ft/sec [✓ (B) 2.667 ft/sec] (C) 3.75 ft/sec (D) 6 ft/sec (E) 10 ft/sec

$$\frac{dx_1}{dt} = 4 \text{ ft/sec}$$

Find $\frac{dx_2}{dt}$

$$\frac{15}{6} = \frac{x_1+x_2}{x_2}$$

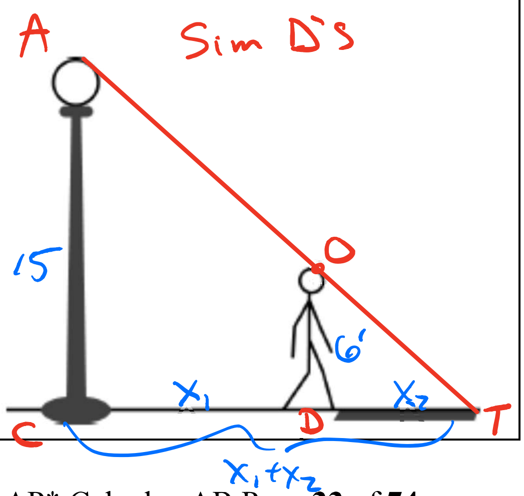
$$15x_2 = 6x_1 + 6x_2$$

$$9x_2 = 6x_1$$

$$\frac{d}{dt}9x_2 = \frac{d}{dt}6x_1$$

$$9\cdot\frac{dx_2}{dt} = 6\cdot\frac{dx_1}{dt}$$

$$\frac{dx_2}{dt} = \frac{2}{3}\frac{dx_1}{dt} = \frac{2}{3}(4) = 8/3$$



10. An ice sculpture in the form of a sphere melts in such a way that it maintains its spherical shape. The volume of the sphere is decreasing at a constant rate of 2π cubic meters per hour. At what rate, in square meters per hour, is the surface area of the sphere decreasing at the moment when the radius is 5 meters? (Note: For a sphere of radius r, the surface area is $4\pi r^2$ and the volume is $\frac{4}{3}\pi r^3$.) LO2

[✓ (A) $\frac{4\pi}{5}$]

(B) 40π

(C) $80\pi^2$

(D) 100π

① $\frac{dV}{dt} = -2\pi \ m^3/hr$

FIND $\left.\frac{dA}{dt}\right|_{r=5}$

② $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$

$\left.\frac{dV}{dt}\right|_{r=5} = 4\pi(5)^2 \cdot \frac{dr}{dt}$

$-2\pi = 4\pi \cdot 25 \cdot \frac{dr}{dt}$

$-\frac{1}{50} = \frac{dr}{dt}$

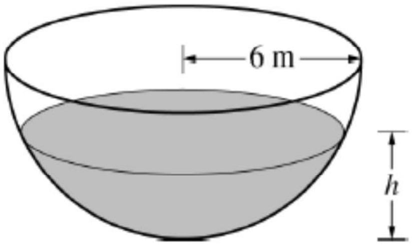
③ $A = 4\pi r^2$

$\frac{dA}{dt} = 8\pi r \cdot \frac{dr}{dt}$

$\left.\frac{dA}{dt}\right|_{r=5} = 8\pi(5) \cdot -\frac{1}{50}$

$\left.\frac{dA}{dt}\right|_{r=5} = -\frac{8\pi}{10}$

11.



A hemispherical water tank, shown above, has a radius of 6 meters and is losing water. The area of the surface of the water is $A = 12\pi h - \pi h^2$ square meters, where h is the depth, in meters, of the water in the tank. When $h = 3$ meters, the depth of the water is decreasing at a rate of $\frac{1}{2}$ meter per minute. At that instant, what is the rate at which the area of the water's surface is decreasing with respect to time?

[✓ (A) 3π square meters per minute]

(B) 6π square meters per minute

(C) 9π square meters per minute

(D) 27π square meters per minute

$\left.\frac{dh}{dt}\right|_{h=3} = -\frac{1}{2} \ m/min$; FIND $\left.\frac{dA}{dt}\right|_{h=3}$

$$\frac{dA}{dt} = 12\pi \cdot \frac{dh}{dt} - 2\pi h \cdot \frac{dh}{dt}$$

$$\left.\frac{dA}{dt}\right|_{h=3} = 12\pi \cdot \left(\frac{-1}{2}\right) - 2\pi(3)\left(-\frac{1}{2}\right)$$

$$= -6\pi + 3\pi$$

$$= -3\pi$$