Name______________________________Date_______________Class__________

Homework 7.3

AP® CALCULUS AB **2001 Question 6**

The function f is differentiable for all real numbers. The point $\left(3, \frac{1}{4}\right)$ is on the graph of $y = f(x)$, and the slope at each point (x, y) on the graph is given by $\frac{dy}{dx} = y^2(6 - 2x)$.

(a) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $\left(3, \frac{1}{4}\right)$.

$$\frac{dy}{dx} = y^2(6-2x)$$

$$\frac{d^2y}{dx^2} = 2y \cdot \frac{dy}{dx}(6-2x) + y^2(-2)$$

$$\frac{d^2y}{dx^2} = 2y \cdot [y^2(6-2x)](6-2x) - 2y^2$$

$$\frac{d^2y}{dx^2} = 2y^3(6-2x)^2 - 2y^2$$

$$\left.\frac{d^2y}{dx^2}\right|_{(3,1/4)} = 2\left(\tfrac{1}{4}\right)^3(6-2(3))^2 - 2\left(\tfrac{1}{4}\right)^2$$

$$= 2\left(\tfrac{1}{64}\right)(6-6)^2 - 2\left(\tfrac{1}{16}\right)$$

$$= 0 - \tfrac{1}{8}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(3,1/4)} = -1/8$$

(b) Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = y^2(6 - 2x)$ with the initial condition $f(3) = \frac{1}{4}$.

$$\int y^{-2}\,dy = \int (6-2x)\,dx$$

$$-y^{-1} = 6x - x^2 + C$$

$$-\frac{1}{y} = -x^2 + 6x + C$$

at $\left(3, \frac{1}{4}\right)$

$$\frac{-1}{\frac{1}{4}} = -(3)^2 + 6(3) + C$$

$$-4 = -9 + 18 + C$$

$$-4 = 9 + C$$

$$-13 = C$$

$$-\frac{1}{y} = -x^2 + 6x - 13$$

$$-1 = (-x^2 + 6x - 13)\,y$$

$$\frac{-1}{-x^2 + 6x - 13} = y$$

$$f(x) = \frac{1}{x^2 - 6x + 13}$$

Consider the differential equation $\frac{dy}{dx} = \frac{3-x}{y}$.

(a) Let $y = f(x)$ be the particular solution to the given differential equation for $1 < x < 5$ such that the line $y = -2$ is tangent to the graph of f. $\boxed{\text{Find the } x\text{-coordinate}}$ of the point of tangency, and determine whether f has a local maximum, local minimum, or neither at this point. Justify your answer.

If $y=-2$ is tangent, then $\frac{dy}{dx}=0$ (horizontal tangent line)

$$\frac{dy}{dx} = \frac{3-x}{y}$$

$$0 = \frac{3-x}{-2}$$

$$0 = 3-x$$

$$x = 3$$

POT $(3,-2)$

2nd Derivative Test

$$\frac{d^2y}{dx^2} = \frac{(-1)(y) - (3-x)(1)\frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-y - (3-x)\left(\frac{3-x}{y}\right)}{y^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(3,-2)} = \frac{-(-2) - (3-3)\left(\frac{3-3}{-2}\right)}{(-2)^2}$$

$$= \frac{2-0}{4}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(3,-2)} = \frac{1}{2}$$

Since $\frac{d^2y}{dx^2} > 0$ at $(3,-2)$ $f(x)$ is concave up which means $(3,-2)$ is a relative minimum of $f(x)$.

(b) Let $y = g(x)$ be the particular solution to the given differential equation for $-2 < x < 8$, with the initial condition $g(6) = -4$. Find $y = g(x)$.

$$\frac{dy}{dx} = \frac{3-x}{y}$$

$$\int y\,dy = \int (3-x)\,dx$$

$$\frac{1}{2}y^2 = 3x - \frac{1}{2}x^2 + C$$

at $(6,-4)$

$$\frac{1}{2}(-4)^2 = 3(6) - \frac{1}{2}(6)^2 + C$$

$$\frac{1}{2}(16) = 18 - \frac{1}{2}(36) + C$$

$$8 = 18 - 18 + C$$

$$8 = C$$

$$\frac{1}{2}y^2 = 3x - \frac{1}{2}x^2 + 8$$

$$y^2 = 6x - x^2 + 16$$

$$y = \pm\sqrt{6x - x^2 + 16}$$

which one contains $(6,-4)$?

$$y = -\sqrt{-x^2 + 6x + 16}$$

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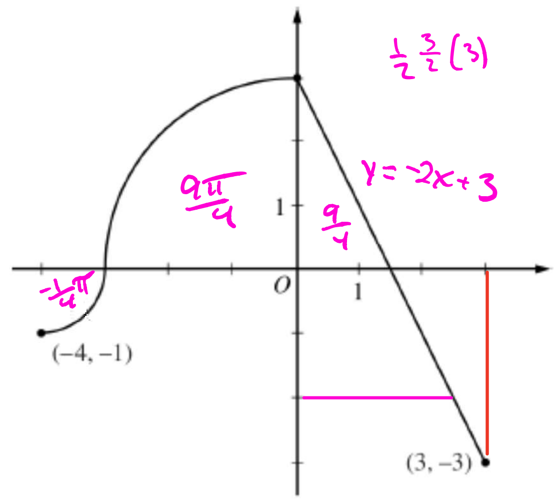
Question 4



Graph of f

The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above.

Let $g(x) = 2x + \int_0^x f(t)\,dt$.

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

Area

$g(-3) = 2(-3) + \int_0^{-3} f(t)dt$

$= -6 - \int_{-3}^{0} f(t)dt$

$= -6 - \left(\frac{1}{4}\pi(3)^2\right)$

$= -6 - \left(\frac{1}{4}\pi\, 9\right)$

$g(-3) = -6 - \frac{9}{4}\pi$

y-value

$g'(x) = 2 + f(x)\cdot x'$

$g'(x) = 2 + f(x)$

$g'(-3) = 2 + f(-3)$

$= 2 + 0$

$g'(-3) = 2$

(b) Determine the x-coordinate of the point at which g has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

EVT

CV

$g'(x) = 2 + f(x)$

$0 = 2 + f(x)$

$f(x) = -2$

at $f(x) = -2$, $f(x) = -2x+3$

$-2 = -2x + 3$

$-5 = -2x$

$\frac{5}{2} = x$

EV $g(-4) = 2(-4) - \int_{-4}^{0} f(t)dt = -8 - \left(-\frac{\pi}{4} + \frac{9\pi}{4}\right) = -8 - 2\pi$

CV $g(5/2) = 2(5/2) + \int_0^{5/2} f(t)dt = 5 + \left(\frac{9}{4} - 1\right) = 5 + \frac{5}{4} = 6.25$

EV $g(3) = 2(3) + \int_0^{3} f(t)dt = 6 + \left(\frac{9}{4} - \frac{1}{2}\cdot\frac{3}{2}(3)\right) = 6 + (0) = 6$

g has an absolute maximum at $x = 5/2$

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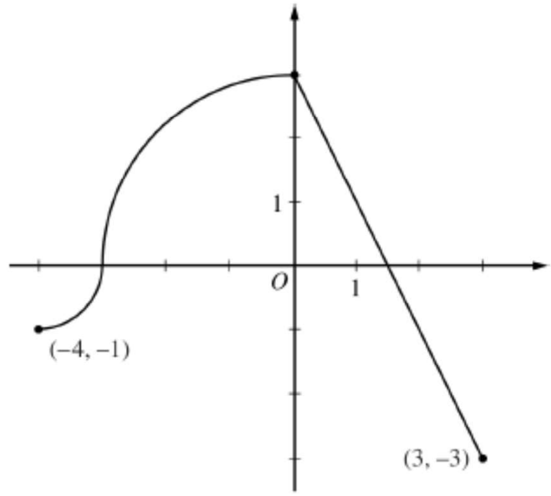
Let $g(x) = 2x + \int_0^x f(t)\,dt$.



Graph of f

(c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.

$g'(x) = 2 + f(x)$

$g''(x) = f'(x)$

$g''(x) > 0$ on $(-4, 0)$ and $g''(x) < 0$ on $(0, 3)$,

thus $g''(x) = f'(x)$ changes sign at $x = 0$.

$\therefore$ $g(x)$ has a point of inflection at $x = 0$.

(d) Find the average rate of change of f on the interval $-4 \le x \le 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

$$\boxed{\begin{aligned} \text{A.R.C.} &= \frac{f(3) - f(-4)}{3 - (-4)} \\ &= \frac{-3 - (-1)}{7} \\ \text{A.R.C.} &= \frac{-2}{7} \end{aligned}}$$

The M.V.T. cannot apply if $f(x)$ is not differentiable for each value on $(-4, 3)$.

$f(x)$ is not differentiable at $x = -3$ and $x = 0$.