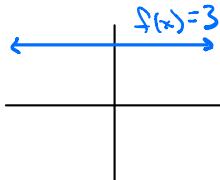


Notes 2.3 – Finding Derivatives Analytically

Polynomials, Sines and Cosines

Consider the function $f(x) = 3$. What does the graph of this function look like? What is the average rate of change of $f(x) = 3$? What is the instantaneous rate of change of $f(x) = 3$?



Based on this thought process, if $f(x) = c$, where c is any constant, then $f'(x) = \underline{\hspace{2cm}}$

Shown below are 6 different polynomial, or polynomial-type, functions. Watch as I find the derivative of each function. See if you can figure out the algorithm that I am using for each function.

$$f(x) = 3x^2 - 2x + 3$$

$$f' = 6x - 2$$

$$f(x) = -5x^3 + 2x^2 - 3x + 1$$

$$\frac{dy}{dx} = -15x^2 + 4x - 3$$

$$f(x) = 6 - 3x^3 + 6x^4$$

$$f' = -9x^2 + 24x^3$$

$$f(x) = -2x^{-1} + 3x^{-2}$$

$$\frac{df}{dx} = 2x^{-2} - 6x^{-3}$$

$$f(x) = 6x^{\frac{2}{3}} + 4x^{-2}$$

$$f' = 4x^{-\frac{1}{3}} - 8x^{-3}$$

$$f(x) = -6x^{-\frac{1}{2}} + 3x^{\frac{1}{2}}$$

$$f' = 3x^{-\frac{3}{2}} + \frac{3}{2}x^{-\frac{1}{2}}$$

Based on what you have seen in the table above, you should now be able to infer how to complete the following Power Rule for Differentiation.

$$\frac{d}{dx}[x^n] = \underline{\hspace{2cm}}$$

Power Rule for Differentiation

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

In order to apply the Power Rule for Differentiation, the equation must be written in “polynomial form.”

Polynomial form

$$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \dots + dx + e$$

Find $f'(x)$ for each of the following functions. Leave your answers with no negative or rational exponents and as single rational functions, when applicable.

$$1. f(x) = \frac{2}{x^2} - 4x^3$$

$$f(x) = 2x^{-2} - 4x^3$$

$$\begin{aligned} f'(x) &= -4x^{-3} - 12x^2 \\ &= \frac{-4}{x^3} - 12x^2 \\ &= \frac{-4}{x^3} - \frac{12x^5}{x^3} \end{aligned}$$

$$f'(x) = \frac{-4 - 12x^5}{x^3}$$

$$2. f(x) = \frac{3x^4 - 3x^2 - 2x}{x}$$

$$f(x) = \frac{x(3x^3 - 3x - 2)}{x}$$

$$f(x) = 3x^3 - 3x - 2$$

$$f'(x) = 9x^2 - 3$$

$$3. f(x) = (x+3)(x+2)(2x+1)$$

$$f(x) = (x^2 + 5x + 6)(2x+1)$$

$$f(x) = 2x^3 + 10x^2 + 12x + x^2 + 5x + 6$$

$$f(x) = 2x^3 + 11x^2 + 17x + 6$$

$$\frac{dy}{dx} = 6x^2 + 22x + 17$$

$$4. f(x) = \frac{x^3 - 5x^2}{x^5}$$

$$\begin{aligned} f(x) &= \frac{x^3}{x^5} - \frac{5x^2}{x^5} \\ f(x) &= x^{-2} - 5x^{-3} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} f(x) &= -2x^{-3} + 15x^{-4} \\ &= -\frac{2}{x^3} + \frac{15}{x^4} \\ \frac{d}{dx} f(x) &= \frac{-2x + 15}{x^4} \end{aligned}$$

$$5. f(x) = \frac{3x}{\sqrt[3]{x^2}}$$

$$f(x) = \frac{3x}{x^{2/3}}$$

$$f(x) = 3x^{1/3}$$

$$\frac{df}{dx} = x^{-2/3}$$

$$\frac{df}{dx} = \frac{1}{\sqrt[3]{x^2}}$$

$$6. f(x) = -4x^{3/4} + 2x^{1/4}$$

$$f'(x) = -3x^{-1/4} + \frac{1}{2}x^{-3/4}$$

$$= \frac{-3\sqrt[4]{x}}{\sqrt[4]{x^2}} + \frac{1}{2\sqrt[4]{x^3}}$$

$$f'(x) = \frac{-6\sqrt[4]{x^2} + 1}{4\sqrt[4]{x^3}}$$

Remember two trigonometric identities that we will use to find the derivatives of the sine and cosine functions.

$$\cos(a+b) = \underline{\cos a \cos b - \sin a \sin b}$$

$$\sin(a+b) = \underline{\sin a \cos b + \sin b \cos a}$$

Use $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ to find $f'(x)$ for each of the following functions. Your results will show the derivative of the sine and cosine functions.

$$7. f(x) = \sin x$$

$$\begin{aligned} f'(x) = (\sin x)' &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cosh h + \sinh \cosh x - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cosh h - \sin x + \sinh \cosh x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x (\cosh h - 1) + \sinh \cosh x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cdot \frac{\cosh h - 1}{h} + \frac{\sinh \cosh x}{h}}{h} \\ &= \sin x \cdot 0 + 1 \cdot \cos x \\ (\sin x)' &= \cos x \end{aligned}$$

$$f' = (\cos x)' = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cos x \cosh h - \sin x \sinh h - \cos x}{h}$$

$$8. f(x) = \cos x$$

$$= \lim_{h \rightarrow 0} \frac{\cos x \cosh h - \cos x - \sin x \sinh h}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cos x (\cosh h - 1) - \sin x \sinh h}{h}$$

$$= \lim_{h \rightarrow 0} \cos x \cdot \frac{\cosh h - 1}{h} - \sin x \cdot \frac{\sinh h}{h}$$

$$= \cos x \cdot (0) - \sin x (1)$$

$$(\cos x)' = -\sin x$$

Differentiation for sine and cosine

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

For each of the following functions, find the equation of the tangent line to the graph of the function at the given point.

9. $f(x) = (2x-1)(x+1)^2$ when $x = -1$

$$\begin{aligned}f(x) &= (2x-1)(x^2+2x+1) \\f(x) &= 2x^3 + 4x^2 + 2x - x^2 - 2x - 1 \\f(x) &= 2x^3 + 3x^2 - 1 \\f'(x) &= 6x^2 + 6x\end{aligned}$$

PoT $(-1, 0)$

$$\begin{aligned}f(x) &= (2x-1)(x+1)^2 \\f(-1) &= [2(-1)-1][(-1)+1]^2 \\&= [-3][0]^2 \\&= [-3][0] \\f(-1) &= 0\end{aligned}$$

SoT $m=0$

$$\begin{aligned}f'(x) &= 6x^2 + 6x + 2 \\f'(-1) &= 6(-1)^2 + 6(-1) \\&= 6(1) - 6 \\&= 6 - 6 \\f'(-1) &= 0\end{aligned}$$

10. $f(\theta) = 4 \sin \theta - \theta$ when $\theta = \frac{\pi}{2}$ O

$$f'(\theta) = 4 \cos \theta - 1$$

PoT

$$\begin{aligned}f(\theta) &= 4 \sin \theta - \theta \\f(0) &= 4 \sin(0) - 0 \\&= 4(0) - 0 \\f(0) &= 0\end{aligned}$$

SoT

$$\begin{aligned}f'(\theta) &= 4 \cos \theta - 1 \\f'(0) &= 4 \cos(0) - 1 \\&= 4 \cdot 1 - 1 \\&= 4 - 1 \\f'(0) &= 3\end{aligned}$$

Tangent Line

$$y - 0 = 3(x - 0)$$

Tangent line

$$y - y_1 = m(x - x_1)$$

$$y - 0 = 3(x + 1)$$

11. $g(\theta) = 2 + 3 \cos \theta$ when $\theta = \pi$

$$g'(\theta) = 0 + 3(-\sin \theta)$$

$$g'(\theta) = -3 \sin \theta$$

PoT

$$\begin{aligned} g(\theta) &= 2 + 3 \cos \theta \\ g(\pi) &= 2 + 3 \cos(\pi) \\ &= 2 + 3(-1) \\ &= 2 - 3 \\ g(\pi) &= -1 \end{aligned}$$

SoT

$$\begin{aligned} g'(\theta) &= -3 \sin \theta \\ g'(\pi) &= -3 \sin(\pi) \\ &= -3(0) \\ g'(\pi) &= 0 \end{aligned}$$

Tangent line
 $y + 1 = 0(x - \pi)$

12. $h(x) = \frac{2x}{\sqrt{x^3}}$ when $x = 2$

$$\begin{aligned} h(x) &= 2x \cdot x^{-\frac{3}{2}} = 2x^{-\frac{1}{2}} \\ h'(x) &= -x^{-\frac{3}{2}} \\ h'(x) &= \frac{-1}{\sqrt{x^3}} \end{aligned}$$

PoT

$$\begin{aligned} h(x) &= \frac{2x}{\sqrt{x^3}} \\ h(2) &= \frac{2(2)}{\sqrt{2^3}} \\ h(2) &= \frac{4}{\sqrt{8}} \end{aligned}$$

SoT

$$\begin{aligned} h'(x) &= \frac{-1}{\sqrt{x^3}} \\ h'(2) &= \frac{-1}{\sqrt{2^3}} \\ &= \frac{-1}{\sqrt{8}} \end{aligned}$$

Tangent line

$$y - \frac{4}{\sqrt{8}} = \frac{-1}{\sqrt{8}}(x - 2)$$

Given the equation of a function, how might you determine the value(s) at which the function has a horizontal tangent? Explain your reasoning.

When $f' = 0$, the tangent line will be horizontal. This is because horizontal lines have slope of zero.

At what value(s) of x will the function $f(x) = x^3 + x$ have a horizontal tangent?

$$f'(x) = 3x^2 + 1$$

$$0 = 3x^2 + 1$$

$$-1 = 3x^2$$

$$\frac{-1}{3} = x^2$$

$$\pm \sqrt{\frac{-1}{3}} = x$$

$$DNE = x$$

There is no value of x that gives a horizontal tangent line to $f(x)$.

At what value(s) of θ at which the function $f(\theta) = \theta + \sin \theta$ has a horizontal tangent on the interval $[0, 2\pi]$?

$$f'(\theta) = 1 + \cos \theta$$

$$0 = 1 + \cos \theta$$

$$-1 = \cos \theta$$

$$\theta = \pi + 2\pi k, \text{ where } k \in \mathbb{Z}$$

when $\theta = \pi$, $f(\theta)$ will have a horizontal tangent.