

### Rules of Differentiation 3.5

#### *Finding Derivatives of Exponential and Logarithm Functions with Bases Other Than e*

##### Differentiation Rule for Exponential Functions

$$\text{Given } a \in \mathbb{R}: \quad \frac{d}{dx} a^{f(x)} = a^{f(x)} \cdot f'(x) \cdot \ln a$$

If above is how we find the derivative of an exponential function, then why did we not do this with the natural logarithm functions whose bases were  $e$ ? Find the derivatives in the boxes below. What do you notice?

Find  $f'(x)$  applying the rule from the previous lesson if  $f(x) = e^{\cos 2x}$ .

$$\begin{aligned} f'(x) &= e^{\cos 2x} \cdot (-\sin(2x)) \cdot 2 \\ &= -2e^{\cos(2x)} \sin(2x) \end{aligned}$$

Find  $f'(x)$  applying the rule above if

$$\begin{aligned} f(x) &= e^{\cos 2x} \\ f'(x) &= e^{\cos 2x} \cdot (-\sin(2x))(2) \cdot \ln e \\ &= -2e^{\cos 2x} \sin(2x) \cdot 1 \end{aligned}$$

Find the derivative of each exponential function given below.

$$f(x) = 4^{x^2}$$

$$\begin{aligned} \dot{f} &= 4^{x^2} \cdot 2x \cdot \ln 4 \\ &= 2x \ln 4 \cdot 4^{x^2} \end{aligned}$$

$$g(x) = x^2 2^{2x}$$

$$\begin{aligned} \text{Product Rule} \\ g'(x) &= 2x \cdot 2^{2x} + x^2 \cdot 2^{2x} (2) \ln 2 \\ g'(x) &= 2x 2^{2x} (1 + x \ln 2) \end{aligned}$$

$$h(x) = 5^{2x-3}$$

$$\begin{aligned} \dot{h} &= 5^{2x-3} \cdot (2) \cdot \ln 5 \\ \dot{h} &= 2 \ln 5 \cdot 5^{2x-3} \end{aligned}$$

$$p(x) = x(3^{-2x})$$

$$\begin{aligned} \text{Product Rule} \\ p'(x) &= 1 \cdot (3^{-2x}) + x(3^{-2x}) \cdot (-2) \cdot \ln 3 \\ &= 3^{-2x} [1 - 2x \ln 3] \end{aligned}$$

$$f(\theta) = 2^{-\theta} \cos(\pi\theta)$$

Product Rule

$(a^{f(\theta)})'$       Chain

$$\frac{d}{d\theta} f(\theta) = 2^{-\theta}(-1) \cdot \ln 2 \cdot \cos(\pi\theta) + 2^{-\theta}(-\sin(\pi\theta)) \cdot (\pi)$$

$$\frac{d}{d\theta} f(\theta) = -2^{-\theta} [\ln 2 \cdot \cos(\pi\theta) + \pi \sin(\pi\theta)]$$

$$f(x) = \frac{3^{5x}}{x}$$

$$f'(x) = \frac{(3^{5x})' \cdot x - 3^{5x} \cdot (1)}{x^2}$$

Quotient Rule

$$= \frac{3^{5x} (5x \ln 3 - 1)}{x^2}$$

$$h(x) = \frac{x^2}{2^{3x}}$$

$$h'(x) = \frac{(2x)2^{3x} - x^2(2^{3x} \cdot 3 \cdot \ln 2)}{(2^{3x})^2}$$

Quotient Rule

$$h'(x) = \frac{x \cdot 2^{3x} (2 - 3x \ln 2)}{2^{6x^2}}$$

For what value(s) of  $x$  would the graph of  $p(x) = 2^{-3x}$  have a normal line whose slope is 2. Show your work and explain your reasoning.

$$S_{ON} = 2 \therefore S_{OT} = -\frac{1}{2}$$

$$p'(x) = 2^{-3x} \cdot (-3) \cdot \ln 2$$

$$-\frac{1}{2} = -3 \ln 2 \cdot 2^{-3x}$$

## Differentiation Rule for Logarithm Functions

$$[\log_a f(x)]' = \frac{f'(x)}{f(x)} \cdot \frac{1}{\ln a}$$

Let's see the same relationship that we saw between derivatives of natural exponential functions and those that have bases other than  $e$  in the derivatives of natural logarithm functions and logarithm functions that have bases other than  $e$ .

Find  $f'(x)$  applying the rule from the previous lesson if  $f(x) = \ln(\sin 3x)$ .

$$\begin{aligned} f'(x) &= \frac{\cos(3x) \cdot 3}{\sin(3x)} \quad \left. \right\} \ln f(x) \\ &= 3 \cot(3x) \end{aligned}$$

Find  $f'(x)$  applying the rule above if  $f(x) = \ln(\sin 3x)$ .

$$\begin{aligned} f'(x) &= \frac{\cos(3x) \cdot 3}{\sin(3x) \cdot \ln e} \quad \left. \right\} \log_a f(x) \\ f'(x) &= 3 \cot(3x) \end{aligned}$$

Find the derivative of each exponential function given below.

$$f(x) = \log_2 4x$$

$$f'(x) = \frac{4}{4x \cdot \ln 2}$$

$$f'(x) = \frac{1}{x \ln 2}$$

$$g(x) = \log_3(\sin x)$$

$$g'(x) = \frac{\cos x}{\sin x \cdot \ln 3}$$

$$g'(x) = \frac{\cot x}{\ln 3}$$

$$h(x) = \log_5(\cos 3x)$$

$$h'(x) = \frac{-\sin(3x) \cdot 3}{\cos(3x) \ln 5}$$

$$h'(x) = \frac{-3 \tan(3x)}{\ln 5}$$

$$p(x) = \log(2^{3x}) = 3x \cdot \log 2$$

$$p(x) = 3 \log 2 \cdot x$$

$$\begin{aligned} p'(x) &= 3 \log 2 \\ &= \log 8 \end{aligned}$$

$$p'(x) = \frac{2^{3x} \cdot 3 \cdot \ln 2}{2^{3x} \ln 10}$$

$$p'(x) = \frac{3 \ln 2}{\ln 10}$$

$$p'(x) = \frac{\ln 8}{\ln 10}$$

$$k(x) = \log_7 \sqrt{x^2 - 1}$$

$$k'(x) = \frac{1}{2} \log_7 (x^2 - 1)$$

$$k'(x) = \frac{1}{2} \frac{(2x)}{(x^2 - 1) \ln 7}$$

$$k'(x) = \frac{x}{(x^2 - 1) \ln 7}$$

$$q(x) = \log_3 \left( \frac{3x}{x-1} \right)$$

$$q'(x) = \log_3 3x - \log_3 (x-1)$$

$$q'(x) = \frac{3}{3x \cdot \ln 3} - \frac{1}{(x-1) \ln 3}$$

$$= \frac{1}{x \ln 3 (x-1)} - \frac{1}{(x-1) \ln 3} \frac{x}{x}$$

$$= \frac{x-1-x}{x \ln 3 (x-1)}$$

$$q'(x) = \frac{-1}{x \ln 3 (x-1)}$$

$$p(x) = \log_2 (\sin x^2)$$

$$p'(x) = \frac{\cos(x^2) \cdot 2x}{\sin(x^2) \cdot \ln 2}$$

$$p'(x) = \frac{2x \cot(x^2)}{\ln 2}$$

$$\frac{3(x-1) - 3x(1)}{(x-1)^2}$$

$$q'(x) = \frac{\frac{3x-3-3x}{(x-1)^2}}{\frac{3x}{x-1} \cdot \ln 3}$$

$$= \frac{\cancel{3x-3-3x}}{\cancel{x-1}} \frac{(x-1)^2}{\cancel{3x \ln 3}}$$

$$= \frac{-3}{3x \ln 3 (x-1)}$$

$$= \frac{-1}{x \ln 3 (x-1)}$$

**Calculator**

For what value(s) of  $x$  will the graph of  $h(x) = \log_2 \left( \frac{x}{2x+1} \right)$  have a tangent line whose slope is  $\frac{5}{2}$ ?

Show your work and explain your reasoning.

$$\text{SoT: } m = \frac{5}{2}$$

$$h(x) = \log_2 x - \log_2 (2x+1)$$

$$h'(x) = \frac{1}{x \ln 2} \frac{(2x+1)}{(2x+1)} - \frac{2}{(2x+1) \ln 2} \cdot \frac{x}{x}$$

$$\frac{5}{2} = \frac{2x+1 - 2x}{x \ln 2 (2x+1)}$$

$$\frac{5}{2} = \frac{1}{x \ln 2 (2x+1)}$$

$$5x \ln 2 (2x+1) = 2$$

$$x \approx -0.842, 0.342$$

The graph of  $h(x)$  will have a tangent line whose slope is  $\frac{5}{2}$  at  $x \approx -0.842$  and  $0.342$  because the derivative of  $h(x)$  represents the slope of the curve at any given point and is the same as the slope of the tangent.

