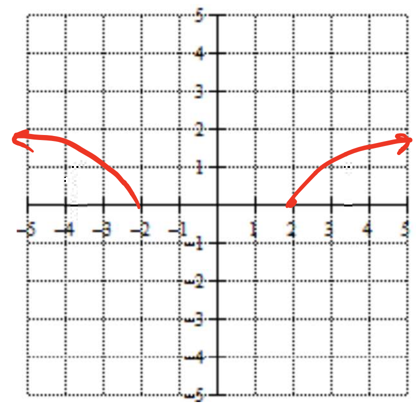
Notes 3.7 – The Relationship between Continuity and Differentiability

In this lesson, our goal is to establish a relationship between a function being continuous at a value of x and a function being differentiable at the same value. In other words, if a function is continuous at a particular value of x, does that imply that it is also differentiable? Or, if a function is differentiable, does that mean that it must also be continuous? Let's investigate three functions.

Consider the function $f(x)=\sqrt{x^2-4}$ at $x = 2$. Answer the questions that follow.

On the grid to the right, sketch a graph of $f(x)$ from your graphing calculator.



Based on the graph, is $f(x)$ continuous at $x = 2$? Explain your reasoning.

I. $f(2)=0 \therefore f(2)$ is defined

II. $\lim\limits_{x\to 2^-} f(x) \neq \lim\limits_{x\to 2^+} f(x) \therefore \lim\limits_{x\to 2} f(x) = \text{DNE}$

$\therefore f(x)$ is not continuous at $x=2$

Find the value of $f'(2)$ to determine if $f(x)$ is differentiable at $x = 2$.

$f(x) = (x^2-4)^{1/2}$

$f'(x) = \frac{1}{2}(x^2-4)^{-1/2}(2x)$

$f'(x) = \dfrac{x}{\sqrt{x^2-4}}$

$f'(2) = \dfrac{2}{\sqrt{(2)^2-4}}$

$= \dfrac{2}{\sqrt{4-4}}$

$= \dfrac{2}{\sqrt{0}}$

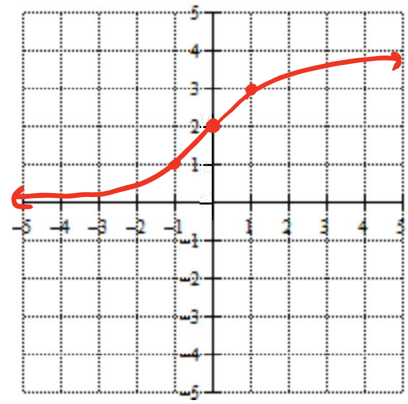
$= \dfrac{2}{0}$

$f'(2) = \text{DNE}$

$\therefore f(x)$ is not differentiable at $x=2$ b/c $f'(2)$ is undefined

Consider the function $f(x)=x^{\frac{1}{3}}+2$ at $x = 0$. Answer the questions that follow.

On the grid to the right, sketch a graph of $f(x)$ from your graphing calculator.



Based on the graph, if $f(x)$ continuous at $x = 0$? Explain your reasoning.

I. $f(0)=2$ $\therefore$ $f(0)$ is defined

II $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 2$ $\therefore$ $\lim_{x\to 0} f(x)$ exists

III $f(0) = \lim_{x\to 0} f(x)$, since they both equal 2.

$\therefore$ $f(x)$ is Continuous at $x=0$

Find the value of $f'(0)$ to determine if $f(x)$ is differentiable at $x = 0$.

$f'(x) = \frac{1}{3}x^{-2/3}$

$f'(0) = \frac{1}{3\sqrt[3]{0^2}}$

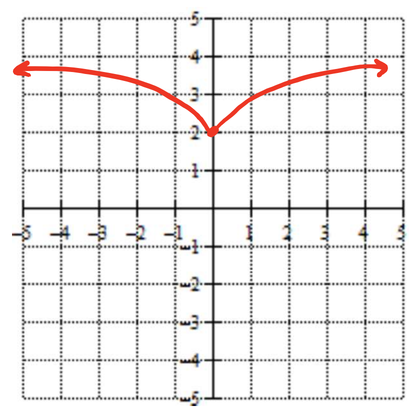
$= \frac{1}{0}$

$f'(0) = DNE$

$f(x)$ is not differentiable at $x=0$ b/c $f'(0)$ is undefined.

Consider the function $f(x)=x^{\frac{2}{3}}+2$ at $x = 0$. Answer the questions that follow.

On the grid to the right, sketch a graph of $f(x)$ from your graphing calculator.



Based on the graph, if $f(x)$ continuous at $x = 0$? Explain your reasoning.

I. $f(0)=2$ $\therefore$ $f(0)$ is defined

II $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 2$ $\therefore$ $\lim_{x\to 0} f(x)$ exists

III $f(0) = \lim_{x\to 0} f(x)$, since they both equal 2.

$\therefore$ $f(x)$ is Continuous at $x=0$

Find the value of $f'(0)$ to determine if $f(x)$ is differentiable at $x = 0$.

$f'(x) = \frac{2}{3}x^{-1/3}$

$f'(0) = \frac{2}{3\sqrt[3]{0}}$

$f'(0) = DNE$

$f(x)$ is not differentiable at $x=0$ b/c $f'(0)$ is undefined.

$-0.842, 0.342$

Based on what we have seen, does continuity imply differentiability or does differentiability imply continuity?

Differentiability implies continuity.

In order for a function to be differentiable at a value of x, then two things must be true:

1. $f(x)$ must be continuous at $x=a$

2. $\lim_{x \to a^-} f'(x) = \lim_{x \to a^+} f'(x)$

Consider the function $g(x)=\begin{cases}\sqrt{x+1}, & 0 \le x \le 3 \\ 5-x, & 3 < x \le 5\end{cases}$ to answer the following questions.

Is $g(x)$ continuous at $x = 3$? Show the complete analysis.

I. $g(3)=2$, $\therefore$ $g(3)$ is defined

II. $\lim_{x \to 3^-} g(x) = 2 = \lim_{x \to 3^+} g(x)$, $\therefore$ $\lim_{x \to 3} g(x)$ exists.

III. $g(3) = \lim_{x \to 3} g(x)$, since they both equal 2.

$\therefore$ $g(x)$ is continuous at $x=3$

Is $g(x)$ differentiable at $x = 3$? Show the complete analysis.

1. $g(x)$ is continuous

2. $\lim_{x \to 3^-} g'(x) = \lim_{x \to 3^-} \frac{1}{2}(x+1)^{-\frac{1}{2}}(1) = \frac{1}{2\sqrt{3+1}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$\lim_{x \to 3^+} g'(x) = \lim_{x \to 3^+} -1 = -1$

$\lim_{x \to 3^-} g'(x) \neq \lim_{x \to 3^+} g'(x)$

$\therefore$ $g(x)$ is not differentiable at $x=3$

For what values of k and m will the function below be both continuous and differentiable at $x = 3$?

$$h(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx+2, & 3 < x \le 5 \end{cases}$$

① Continuous

$\lim_{x\to 3^-} h(x) = \lim_{x\to 3^+} h(x)$

$\lim_{x\to 3^-} k\sqrt{x+1} = \lim_{x\to 3^+} (mx+2)$

$k\sqrt{3+1} = m(3)+2$

$\sqrt{4}\, k = 3m+2$

$2k = 3m+2$

$k = \frac{3}{2}m+1$

② Diff

$\lim_{x\to 3^-} h'(x) = \lim_{x\to 3^+} h'(x)$

$\lim_{x\to 3^-} (k\sqrt{x+1})' = \lim_{x\to 3^+} (mx+2)'$

$\lim_{x\to 3^-} \frac{1}{2}k(x+1)^{-\frac{1}{2}}\cdot(1) = \lim_{x\to 3^+} (m)$

$\frac{k}{2\sqrt{3+1}} = m$

$\frac{k}{2\cdot\sqrt{4}} = m$

$\frac{k}{2\cdot 2} = m$

$k = 4m$

③ $\frac{3}{2}m+1 = 4m$

$3m+2 = 8m$

$2 = 5m$

$\frac{2}{5} = m$

④ $k = 4m$

$k = 4\left(\frac{2}{5}\right)$

$k = \frac{8}{5}$

$k = 8/5$ and $m = 2/5$ to make $h(x)$ continuous and differentiable

For what values of a and b will the function below be differentiable at $x = 1$?

$$f(x) = \begin{cases} 3ax^2 + 2bx + 1, & x \le 1 \\ ax^4 - 4bx^2 - 3x, & x > 1 \end{cases}$$

① Continuous

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$\lim_{x\to 1^-} (3ax^2 + 2bx + 1) = \lim_{x\to 1^+} (ax^4 - 4bx^2 - 3x)$

$3a(1)^2 + 2b(1) + 1 = a(1)^4 - 4b(1)^2 - 3(1)$

$3a + 2b + 1 = a - 4b - 3$

$2a + 2b + 1 = -4b - 3$

$2a = -6b - 4$

$2a + 6b = -4$

② Differentiable

$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^+} f'(x)$

$\lim_{x\to 1^-} (3ax^2 + 2bx + 1)' = \lim_{x\to 1^+} (ax^4 - 4bx^2 - 3x)'$

$\lim_{x\to 1^-} (6ax + 2b) = \lim_{x\to 1^+} (4ax^3 - 8bx - 3)$

$6a(1) + 2b = 4a(1)^3 - 8b(1) - 3$

$6a + 2b = 4a - 8b - 3$

$2a + 10b = -3$

times (-1)

③

$2a + 6b = -4$

$-2a - 10b = 3$

$-4b = -1$

$b = \frac{1}{4}$

④ $2a + 10\left(\frac{1}{4}\right) = -3$

$2a + \frac{5}{2} = -3$

$4a + 5 = -6$

$4a = -11$

$a = -11/4$

$\boxed{a = \frac{-11}{4} \text{ and } b = \frac{1}{4} \text{ to make } f(x) \text{ differentiable at } x = 1}$