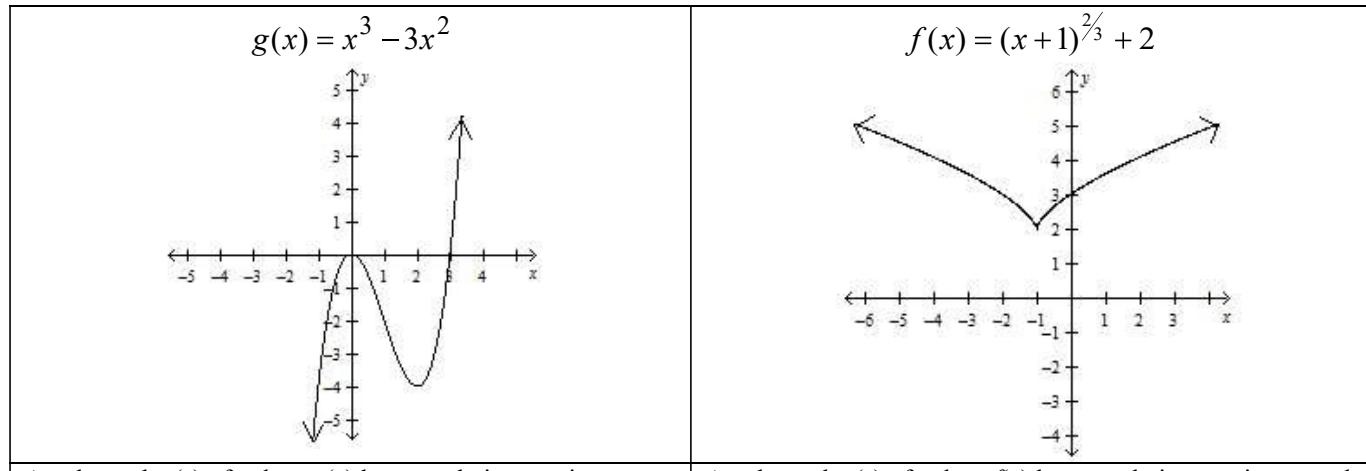


Notes 4.4 – Analytical and Graphical Connections between $f(x)$, $f'(x)$, and $f''(x)$ – Part I

Let's begin by remembering what we learned about the connections between a function, f , and its first derivative, f' , in the context of polynomial functions.

Now that we know how to take the derivative of all types of functions, we have to tweak our understanding of this concept just slightly. Let's think about two different functions for a moment.



At what value(s) of x does $g(x)$ have a relative maximum and a relative minimum?

$$x = 0, 3$$

What can be said about $g'(x)$ at the value(s) indicated above?

$$g'(x) = 0$$

At what value(s) of x does $f(x)$ have a relative maximum and a relative minimum?

$$x = -1$$

What can be said about $f'(x)$ at the value(s) indicated above?

$$f'(x) = \text{undefined}$$

Algebraically find $f'(x)$. Then, perform a sign analysis of $f'(x)$ using any value that makes $f'(x) = 0$ or undefined.

Does this sign analysis establish what is depicted graphically above?

$$f(x) = (x+1)^{2/3} + 2$$

$$\text{CV : } x = -1$$

$$f'(x) = \frac{2}{3}(x+1)^{-1/3}$$

$$\frac{2}{3\sqrt[3]{x+1}}$$

$\cancel{x+1}$

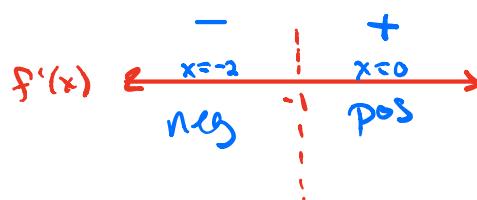
$$0 = \cancel{3\sqrt[3]{x+1}}$$

$$0 = \sqrt[3]{x+1}$$

$$0 = x+1$$

$$x = -1$$

$$f''(x) = \text{undefined at } x = -1$$



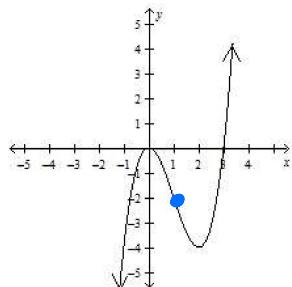
- $f(x)$ is increasing on $(-1, \infty)$ b/c $f' > 0$ on this interval

- $f(x)$ is decreasing $(-\infty, -1)$ b/c $f' < 0$ on this interval

Critical Values – An x-value that changes the nature of a curve. A curve's nature is increasing, decreasing, concave up, concave down, or even constant. Therefore, a critical value can produce a relative minimum point, relative maximum point, inflection point or be a vertical asymptote.

If $F'(x)$...	then $F(x)$...
...is = 0 or is undefined at $x = a$,	has a potential relative max or min
...is > 0 ,	is increasing
...is < 0 ,	is decreasing
...changes from positive to negative,	Rel MAX
...changes from negative to positive,	Rel MIN

The FIRST DERIVATIVE of a function identifies intervals where a function is increasing, decreasing, has a relative maximum or has a relative minimum.



- 1) Given $g(x) = x^3 - 3x^2$, find the relative extrema using a first derivative sign diagram.

CV: 0, 2

① $f'(x) = 3x^2 - 6x$
 $0 = 3x(x-2)$
 ZON ZOD
 $3x=0 \quad x-2=0$
 $x=0 \quad x=2$
 $f'(x)=0 \text{ at } x=0, 2$

② $(-) (-) \quad (+) (+) \quad (+) (+)$
 POS | NEG | POS
 $f'(x) < x=-1 \quad x=1 \quad x=3$
 increasing MAX decreasing MIN increasing

In a similar fashion, the SECOND DERIVATIVE identifies intervals where a function is concave up, concave down, or has a point of inflection.

- 2) Given $g(x) = x^3 - 3x^2$, find the inflection points using a second derivative sign diagram.

CV: $x=1$

③ $f''(x) = 6x-4$
 $0 = 6(x-1)$
 CV: $x=1$
 ZON ZOD
 $x-1=0 \quad x=1$
 $f''(x)=0 \text{ at } x=1$

④ $f''(x) \leftarrow x=0 \quad x=2 \rightarrow$
 NEG | POS
 CCW (1) CCUP
 IP

⑤ FIND CP if needed

If $F''(x) \dots$	then $F(x) \dots$
...is = 0 or is undefined at $x = a$,	has a potential POI
...is > 0,	is concave up
...is < 0,	is concave down
...changes from positive to negative,	(has a POI) Rel MAX
...changes from negative to positive,	(has a POI) Rel MIN

For each function, $f(x)$, determine the x -values of the point(s) of inflection and the interval(s) where the graph of $f(x)$ is concave up or concave down. Justify your answers.

$$f(x) = x^4 - 4x^3$$

$$f'(x) = 4x^3 - 12x^2$$

$$CV: x=0, 2$$

$$f''(x) = 12x^2 - 24x$$

$$0 = 12x(x-2)$$

$$\text{ZON} \quad \text{ZOD}$$

$$\begin{aligned} 0 = 12x \quad & \left. \begin{aligned} x-2 &= 0 \\ x &= 2 \end{aligned} \right\} \\ 0 = x \quad & \left. \begin{aligned} x-2 &= 0 \\ x &= 2 \end{aligned} \right\} \end{aligned}$$

$$f''(x) \neq \text{und}$$

$$f''(x) = 0 \text{ at } x=0, 2$$

$$f(x) = 2xe^x$$

$$f'(x) = 2e^x + 2xe^x$$

$$CV: x=-2$$

$$f''(x) = 2e^x + 2 \cdot e^x + 2xe^x$$

$$f''(x) = 4e^x + 2xe^x$$

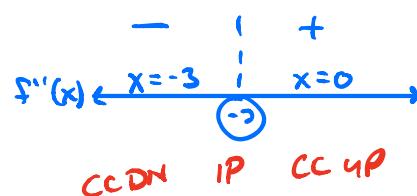
$$0 = 2e^x(2+x)$$

$$\begin{aligned} 0 = 2e^x \quad & \left. \begin{aligned} 2+x &= 0 \\ x &= -2 \end{aligned} \right\} \\ 0 = e^x \quad & \left. \begin{aligned} x &= -2 \end{aligned} \right\} \end{aligned}$$

$$\text{ZON} \quad \text{ZOD}$$

$$\text{No solution} \quad f''(x) = 0 \text{ at } x=-2$$

$$f''(x) \neq \text{und}$$



$$(-)(-) = + \quad (+)(-) = - \quad (-)(+) = +$$

$$f''(x) \leftarrow \begin{cases} x = -1 & \\ x = 1 & \\ x = 3 & \end{cases}$$

$$\text{CC UP} \quad \text{IP} \quad \text{CC DN} \quad \text{IP} \quad \text{CC UP}$$

- $f(x)$ is concave up on $(-\infty, 0) \cup (2, \infty)$ b/c $f''(x)$ is positive on those intervals.
- $f(x)$ is concave down on $(0, 2)$ b/c $f''(x)$ is negative on that interval
- $f(x)$ has an inflection point at $x=0$ and $x=2$ b/c $f''(x)$ changes sign at $x=0$ and $x=2$.

- $f(x)$ is concave up on $(-2, \infty)$ b/c $f''(x)$ is positive on that interval
- $f(x)$ is concave down on $(-\infty, -2)$ b/c $f''(x)$ is negative on that interval
- $f(x)$ has an inflection point at $x=-2$ b/c $f''(x)$ changes sign at $x=-2$.

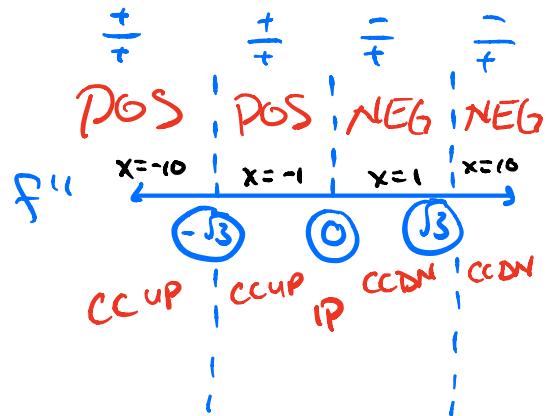
$$f'(x) = \frac{3}{x^2 - 3}$$

CV: $x=0, \pm\sqrt{3}$

$f''(x) = \frac{0 \cdot (x^2 - 3) - 3(2x)}{(x^2 - 3)^2}$	
$f''(x) = \frac{-6x}{(x^2 - 3)^2}$	
ZON	ZOD
$-6x = 0$ $x = 0$	$(x^2 - 3)^2 = 0$ $x^2 - 3 = 0$ $x^2 = 3$ $x = \pm\sqrt{3}$

$f''(x) = 0$ at $x=0$

$f''(x)$ undefined at $x=\pm\sqrt{3}$



- $f(x)$ is concave up on $(-\infty, -\sqrt{3}) \cup (-\sqrt{3}, 0)$ b/c $f''(x)$ is positive on those intervals
- $f(x)$ is concave down on $(0, \sqrt{3})$ b/c $f''(x)$ is negative on that interval
- $f(x)$ has an inflection point at $x=0$ b/c $f''(x)$ changes sign at $x=0$.

While the second derivative is primarily used to determine intervals of concavity and points of inflection, it can also be used to identify relative maximums and minimums of a function.

THE SECOND DERIVATIVE TEST

An Alternate Way to Identify Relative Maximums and Minimums using the 2nd Derivative

IF $x = a$ is a critical value of $f(x)$, then $f'(a) = 0$ or $f'(a) = \text{undefined}$, making $f(a)$ a potential relative maximum or minimum.

- 1) If $f''(a) > 0$, then $f(x)$ is concave up at $x = a \therefore f(a) = \text{relative minimum}$
- 2) If $f''(a) < 0$, then $f(x)$ is concave down at $x = a \therefore f(a) = \text{relative maximum}$
- 3) If $f''(a) = 0$, then The Second Derivative Test is inconclusive

Use the second derivative test to locate and classify all x -values of relative extrema for each function.

1. $f(x) = x^4 - 4x^3 + 2$

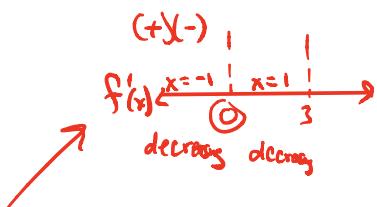
CV: $x=0, 3$	
$f'(x) = 4x^3 - 12x^2$	
$0 = 4x^2(x-3)$	
ZOM	ZOD
$x=0, 3$	$f'(x) \neq \text{end}$
$f'(x) = 0 \text{ at } x=0, 3$	

2nd Derivative Test

$$f''(x) = 12x^2 - 24x$$

$f''(0) = 12(0)(0-2) = 0$, INCONCLUSIVE, $x=0$ is not rel max or min of $f(x)$.

$f''(3) = 12(3)(3-2) = +$, CCUP, $\therefore x=3$ is rel MIN of $f(x)$



2. $f(\theta) = 2\cos\theta - \theta$, on the interval $0 < \theta < 2\pi$

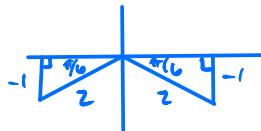
$$CV: \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$



$$f'(\theta) = -2\sin\theta - 1$$

ZOM	ZOD
$0 = -2\sin\theta - 1$	$f''(\theta) \neq \text{end}$
$1 = -2\sin\theta$	
$-\frac{1}{2} = \sin\theta$	
$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$	

$$f''(\theta) = 0 \text{ at } \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$



2nd Derivative Test

$$f''(\theta) = -2\cos\theta$$

$f''\left(\frac{7\pi}{6}\right) = -2\cos\left(\frac{7\pi}{6}\right) > 0$, CCUP $\therefore \text{MIN at } \theta = -\frac{7\pi}{6}$

$f''\left(\frac{11\pi}{6}\right) = -2\cos\left(\frac{11\pi}{6}\right) < 0$, CC DN $\therefore \text{MAX at } \theta = \frac{11\pi}{6}$

