Notes 5.1 – The Extreme Value Theorem

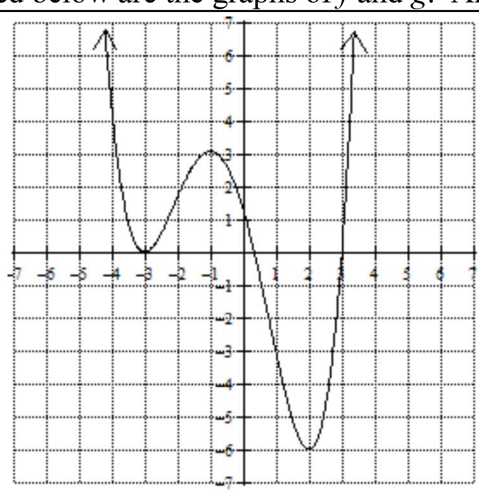
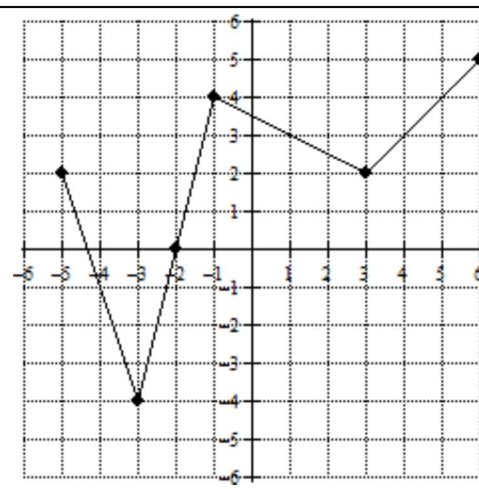
In the previous unit, we investigated heavily how to locate relative extrema of the graph of a function by using the derivative. In pre-calculus, we talked about the difference between relative and absolute extrema. In the space below, distinguish between the two.

Definitions of Relative and Absolute Extrema of a Function

Relative extrema are points on the graph of a function that changes the function from increasing to decreasing or vice versa.

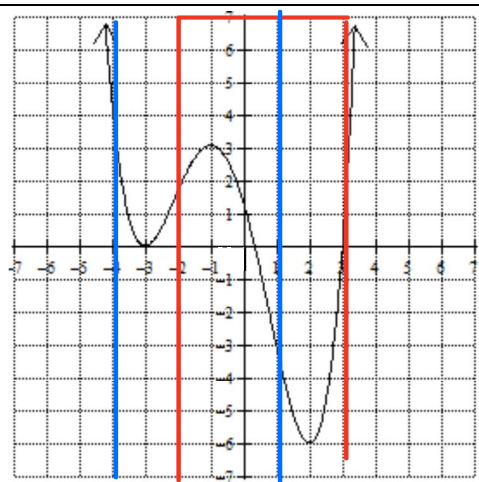
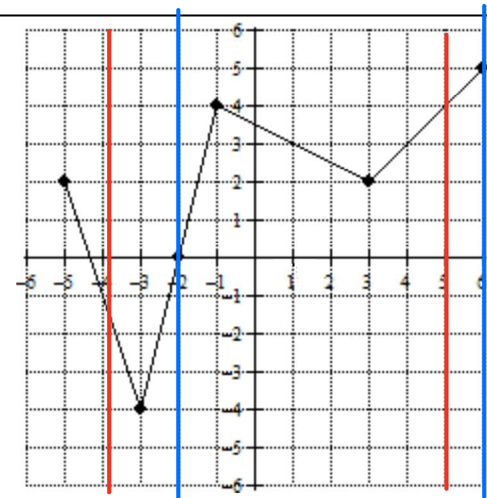
Absolute extrema are the highest and lowest points on the graph of a function.

Pictured below are the graphs of *f* and *g*. Answer the questions about these two functions.

 Graph of $f(x)$	 Graph of $g(x)$
Identify the coordinates of the relative extrema of *f*. Relative min: $(-3,0)$ and $(2,-6)$ Relative max: $(-1,3)$	Identify the coordinates of the relative extrema of *g*. Relative min: $(-3,-4)$ and $(3,2)$ Relative max: $(-1,4)$
On the domain of *f*, what are the coordinates of the absolute extrema of *f*? Absolute max: none Absolute min: $(2,-6)$	On the domain of *g*, what are the coordinates of the absolute extrema of *g*? Absolute max: $(6,5)$ Absolute min: $(-3,-4)$

On the domain of the given function, did the absolute extrema occur at the function's relative extrema?

The absolute extrema could occur at the function's relative extrema, but it doesn't have to.

 Graph of $f(x)$	 Graph of $g(x)$
On the interval $-2 \le x \le 3$, what are the absolute extrema of f? Absolute max: $(-1, 3)$ Absolute min: $(2, -6)$	On the interval $-4 \le x \le 5$, what are the absolute extrema of g? Absolute max: $(-1, 4)$ and $(5, 4)$ Absolute min: $(-3, -4)$
On the interval $-4 \le x \le 1$, what are the absolute extrema of f? Absolute max: $(-4, 4)$ Absolute min: $(1, -3)$	On the interval $-2 \le x \le 6$, what are the absolute extrema of g? Absolute max: $(6, 5)$ Absolute min: $(-2, 0)$

When the domain is restricted to a particular closed interval, at what three places that the absolute extrema could exist?

Absolute extrema could exist...

1. The enpoints of the interval
2. where $f'(x) = 0$
3. where $f'(x) = $ und

The Extreme Value Theorem (E. V. T.):

$f(x)$ has at least one absolute maximum and one absolute minimum on $[a, b]$ if
(a) $f(x)$ is a continuous function on $[a, b]$

$1 - 6 + 9 + 2$

Consider the cubic function $f(x) = -x^3 - 6x^2 - 9x + 2$ to answer the following questions.

a. Determine the intervals where *f* is increasing and decreasing. Justify your answers.

$f'(x) = -3x^2 - 12x - 9$

$0 = -3(x^2 + 4x + 3)$

$0 = -3(x+3)(x+1)$

$f' = -3(x+3)(x+1)$

$f'(x)$: $x=-4$ | -3 | $x=-2$ | -1 | $x=0$

Neg | Pos | Neg

f is increasing on $(-3,-1)$ b/c $f' > 0$

f is decreasing on $(-\infty,-3) \cup (-1,\infty)$ b/c $f' < 0$

b. Determine the coordinates of the relative extrema of *f*. Justify your answers.

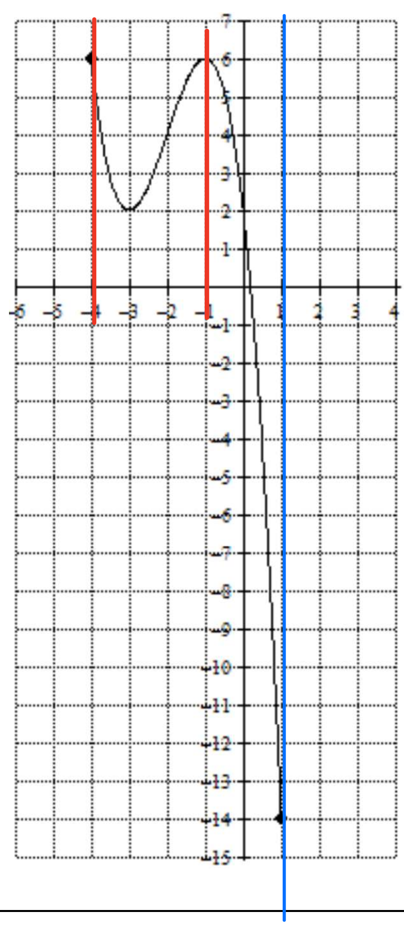
Relative minimum at $x = -3$ b/c f' changes from negative to positive at $x = -3$

$f(-3) = 2$

Relative maximum at $x = -1$ b/c f' changes from positive to negative at $x = -1$

$f(-1) = 6$

Pictured below is a graph of the function *f* on the closed interval $-4 \le x \le 1$.



Identify the absolute maximum of *f* on the closed interval $-4 \le x \le -1$.

$(-4, 6)$ and $(-1, 6)$

Identify the absolute minimum of *f* on the closed interval $-4 \le x \le -1$.

$(-3, 2)$

Identify the absolute maximum of *f* on the closed interval $-4 \le x \le 1$.

$(-4, 6)$ and $(-1, 6)$

Identify the absolute minimum of *f* on the closed interval $-4 \le x \le 1$.

$(1, -14)$

Use the extreme value theorem to locate the absolute extrema of the function $f(x) = -x^3 - 6x^2 - 9x + 2$ on the given closed intervals. Your algebraic results should concur with your graphical conclusions from the previous page.

Interval: $-4 \le x \le -1$	Interval: $-4 \le x \le 1$
CV $f'(x) = -3x^2 - 12x - 9$ $0 = -3(x^2 + 4x + 3)$ $0 = -3(x+3)(x+1)$ $x = -3, -1$ EV $f(-4)$: 6 CV $f(-3)$: 2 EV, CV $f(-1)$: 6 ABS MAX: $(-1, 6)$ and $(-4, 6)$ ABS MIN: $(-3, 2)$	EV $f(-4) = 6$ CV $f(-3) = 2$ CV $f(-1) = 6$ EV $f(1) = -14$ ABS MAX: $(-1, 6)$ and $(-4, 6)$ ABS MIN: $(1, -14)$

For each of the following functions, state specifically why the E. V. T. is or is not applicable on the given interval.

Interval: $-5 \le x \le 0$	
$H(x) = \frac{3x+2}{x+3}$	The E.V.T is not applicable because $H(x)$ is not continuous on $(-5, 0)$ b/c $x \neq -3$
$G(x) = 2x\sqrt{x-3}$	The E.V.T is not applicable because $G(x)$ is not continuous on $(-5, 0)$ b/c $x \geq 3$ to make $G(x)$ defined.
$f(x) = \ln(x+7)$ $x+7 \geq 0$ $x \geq -7$	The E.V.T is applicable because $f(x)$ is continuous on $(-5, 0)$

y-value

Given the functions below, determine the absolute extreme values of the function on the given interval, provided the extreme value theorem is applicable. If it is not, state specifically why it is not.

1. $f(x) = x^3 - 2x^2 - 3x - 2$ on $[-1, 3]$

$f'(x) = 3x^2 - 4x - 3$

$0 = 3x^2 - 4x - 3$

$x \approx -0.535$

$x \approx 1.869$

$\boxed{\begin{array}{l} \text{EV} \quad f(-1) = -2 \\ \text{CV} \quad f(-0.535) \approx -1.121 \\ \text{CV} \quad f(1.869) \approx -8.065 \\ \text{EV} \quad f(3) = -2 \end{array}}$

ABS MAX value: $y = -1.121$

ABS MIN value: $y = -8.065$

2. $g(x) = \sin^2 x + \cos x$ on $\frac{\pi}{2} \le x \le 2\pi$

$g'(x) = 2\sin(x)\cdot\cos(x) - \sin(x)$

$0 = \sin(x)\left[2\cos(x) - 1\right]$

$0 = \sin(x)$

$x = 0, \pi, 2\pi$

Not in Domain

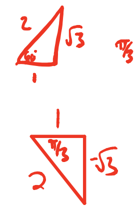
$2\cos(x) - 1 = 0$

$2\cos(x) = 1$

$\cos(x) = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

Not in Domain



EV $g(\frac{\pi}{2}) = 1 + 0 = 1$

CV $g(\pi) = 0 + (-1) = -1$

CV $g(\frac{5\pi}{3}) = (-\frac{\sqrt{3}}{2})^2 + \frac{1}{2} = \frac{3}{4} + \frac{2}{4} = \frac{5}{4}$

EV $g(2\pi) = 1 + 0 = 1$

ABS MAX Value: $y = \frac{5}{4}$

ABS MIN value: $y = -1$

3. $f(x) = (x+2)^{\frac{2}{3}}$ on $[-3, 6]$

$f'(x) = \frac{2}{3}(x+2)^{-\frac{1}{3}}(1)$

$0 = \frac{2}{3\sqrt[3]{x+2}}$

ZOD

$3\sqrt[3]{x+2} = 0$

$\sqrt[3]{x+2} = 0$

$x+2 = 0$

$x = -2$

EV $f(-3) = 1$

CV $f(-2) = 0$

EV $f(6) = 4$

ABS Max value: $y = 4$

ABS MIN value: $y = 0$

4. $h(x) = \ln\left(x^2 - 4\right)$ on $[-1, 3]$

$x^2 - 4 > 0$

$x^2 > 4$

Boundary

$x^2 = 4$

$x = \pm 2$

$+ \mid - \mid +$

$-2 \quad 2$

$-2 \le x \le 2$ are excluded values.

The E.V.T doesn't apply b/c $h(x)$ is undefined on $[-1, 2]$ which is part of the given interval of $[-1, 3]$