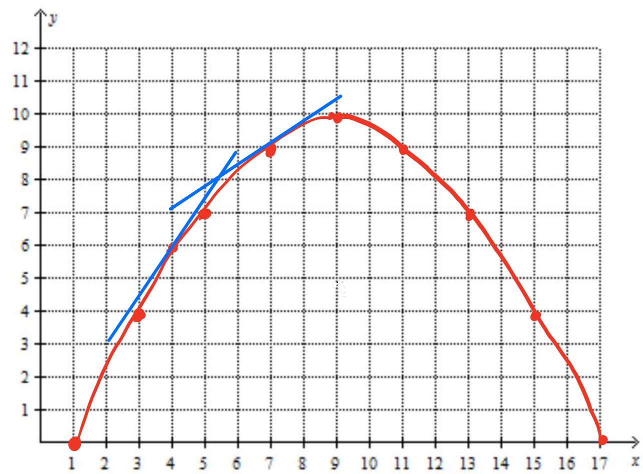
Notes 5.2 – The Derivate as a Rate of Change
Mean Value Theorem and Rolle's Theorem

Consider the values of a differentiable function, $f(x)$, in the table below to answer the questions that follow. Plot the points and connect them on the grid below.

x	0	2	4	6	8	10	12	14	16
$f(x)$	1	5	8	10	11	10	8	5	1



In calculus, the derivative has many interpretations. One of the most important interpretations is that the derivative represents the Rate of Change of a Function. When speaking of rate of change, there are two rates of change that can be found that are associated with a function—average rate of change and instantaneous rate of change.

Average Rate of Change of $f(x)$ on an Interval

On interval $[a, b]$ the average rate of change is

$$A.R.C. = \frac{f(a) - f(b)}{a - b}$$

Instantaneous Rate of Change of $f(x)$ at a Point

At a value $x = a$ the instantaneous rate of change is

$$f'(a)$$

Find the average rate of change of $f(x)$ on the interval [2, 12].

$$ARC = \frac{f(2) - f(12)}{2 - 12}$$

$$= \frac{5 - 8}{-10}$$

$$= \frac{-3}{-10}$$

$$ARC = \frac{3}{10}$$

Is the instantaneous rate of change of f at $x = 4$ greater than the rate of change at $x = 6$? Justify.

Yes, $f'(4) > f'(6)$ b/c the tangent line drawn at $x = 4$ on $f(x)$ is steeper than the tangent line drawn at $x = 6$.

Rolle's Theorem

Rolle's Theorem guarantees a value of c on (a, b) such that $f'(c) = 0$ if
(a) $f(x)$ is continuous on $[a, b]$
(b) $f(x)$ is differentiable on (a, b)
(c) $f(a) = f(b)$

Consider the function, *f*(*x*), presented on the previous page. Does Rolle's Theorem apply on the following intervals? Explain why or why not?

Interval [2, 14]	a) f is continuous on $[2, 14]$ b) differentiable on $(2, 14)$ c) and $f(2) = f(14) = 5$ $\therefore$ Rolle's Theorem guarantees a value of c on $(2, 14)$ such that $f'(c) = 0$ The guaranteed value of c is 8.
Interval [2, 10]	a) f is continuous on $[2, 10]$ b) differentiable on $(2, 10)$ c) $f(2) \neq f(14)$ $\therefore$ Rolle's Theorem does NOT guarantee a value of c on $(2, 10)$ such that $f'(c) = 0$

For each of the functions below, determine whether Rolle's Theorem is applicable or not. Then, apply the theorem to find the values of *c* guaranteed to exist.

1. $g(x) = 9x^2 - x^4$ on the interval $[-3, 0]$

a) $g(x)$ is a polynomial thus continuous on $[-3, 0]$
b) $g(x)$ is a polynomial thus differentiable on $(-3, 0)$
c) $g(-3) = g(0) = 0$.
Rolle's Theorem guarantees a value of c on $(-3, 0)$ such that $f'(c) = 0$

$g'(x) = 18x - 4x^3$
$g'(c) = 0$
$0 = 18c - 4c^3$
$0 = 2c(9 - 2c^2)$
$0 = 2c$ | $9 - 2c^2 = 0$
$0 = c$ | $9 = 2c^2$
$0 \notin (-3, 0)$ | $9/2 = c^2$
$\pm 3/\sqrt{2} = c$
$\frac{3}{\sqrt{2}} \notin (-3, 0)$
$\therefore c = -\sqrt{3}/2$

2. $g(x) = \dfrac{\sin 2x}{x + 2}$ on the interval $[-4, -1]$

a) $g(x)$ is not continuous on $[-4, -1]$ b/c $g(-2)$ is undefined.
$\therefore$ Rolle's Theorem does not guarantee a value of c such that $g'(c) = 0$.

Rolle's Theorem guarantees that if a function is continuous on the closed interval [*a*, *b*], differentiable on the open interval (*a*, *b*), and *f*(*a*) = *f*(*b*), then there is guaranteed to exist a value of *c* on (*a*, *b*) where the **instantaneous rate of change is equal to zero.**

The Mean Value Theorem is similar. In fact, Rolle's Theorem is a specific case of what is known in calculus as the Mean Value Theorem.

Average

Mean Value Theorem

The Mean Value Theroem guarantees $f'(c) = \frac{f(a)-f(b)}{a-b}$ if

(a) $f(x)$ is continuous on $[a, b]$

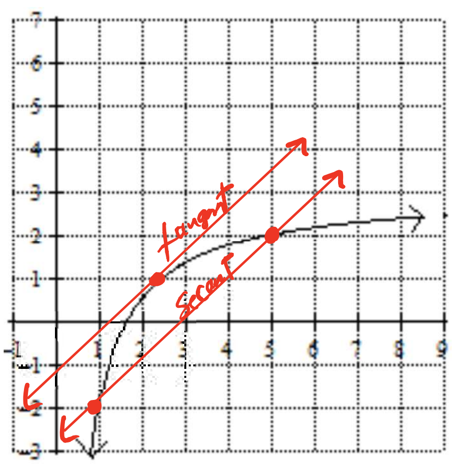
(b) $f(x)$ is differentiable on (a, b)

(ARC = IRC)

Consider the function $h(x) = 3 - \frac{5}{x}$. The graph of *h*(*x*) is pictured below. Does the M.V.T. apply on the interval [–1, 5]? Explain why or why not.

a) $h(x)$ is not continuous on $[-1,5]$ b/c $h(x)$ is not continuous at $x=0$

$\therefore$ The MVT does NOT apply.



Does the M.V.T. apply on the interval [1, 5]? Why or why not?

a) $h(x)$ is continuous on $[1,5]$

b) $h(x)$ is differentiable on $(1,5)$

$\therefore$ The MVT applies

Graphically, what does the M.V.T. guarantee for the function on the interval [1, 5]? Draw this on the graph to the left.

The MVT guarantees a value of $x=c$ where the tangent line drawn to $h(c)$ is parallel to the secant line passing through $(1,-2)$ and $(5,2)$

Apply the M.V.T. to find the value(s) of *c* guaranteed for *h*(*x*) on the interval [1, 5]

$h(x) = 3 - 5x^{-1}$

$h'(x) = 5x^{-2}$

$h'(x) = \frac{5}{x^2}$

$ARC = \frac{h(1)-h(5)}{1-5}$

$= \frac{(-2)-(2)}{-4}$

$= \frac{-4}{-4}$

$ARC = 1$

$h'(c) = ARC$

$\frac{5}{c^2} = 1$

$5 = c^2$

$\pm\sqrt{5} = c$

$-\sqrt{5} \notin [1,5]$, $\therefore$ $c = \sqrt{5}$

Explain why you cannot apply the Mean Value Theorem for $f(x) = x^{\frac{2}{3}} - 2$ on the interval [−1, 1].

a) $f(x)$ is continuous on $[-1,1]$

b) $f'(x) = \frac{2}{3}x^{-1/3}$

$f'(x) = \frac{2}{3\sqrt[3]{x}} \Longrightarrow f'(x)$ is undefined at $x=0$.

$f'(x)$ is not differentiable for all values of x on $[-1,1]$

$\therefore$ The MVT does NOT apply to $f(x)$

Find the equation of the tangent line to the graph of $f(x) = 2x + \sin x + 1$ on the interval (0, π) at the point which is guaranteed by the mean value theorem.

$f'(x) = 2 + \cos x$

$ARC = \frac{f(0) - f(\pi)}{0 - \pi}$

$= \frac{1 - (2\pi + 1)}{-\pi}$

$= \frac{-2\pi}{-\pi}$

$ARC = 2$

$f'(c) = ARC$

$2 + \cos(c) = 2$

$\cos(c) = 0$

$c = \frac{\pi}{2}$

PoT $(\frac{\pi}{2}, \pi + 2)$	SoT: $m = 2$	Tangent line
$f(\frac{\pi}{2}) = 2(\frac{\pi}{2}) + \sin(\frac{\pi}{2}) + 1$	$f'(\frac{\pi}{2}) = 2 + \cos(\frac{\pi}{2})$	$y - (\pi + 2) = 2(x - \frac{\pi}{2})$
$= \pi + 1 + 1$	$= 2 + 0$	
$f(\frac{\pi}{2}) = \pi + 2$	$f'(\frac{\pi}{2}) = 2$	

The Mean Value Theorem guarantees that if a function is continuous on the closed interval [*a*, *b*] and differentiable on the open interval (*a*, *b*), then there is guaranteed to exist a value of *c* where the **instantaneous rate of change at *x* = *c* is equal to the average rate of change of *f* on the interval [*a*, *b*].**