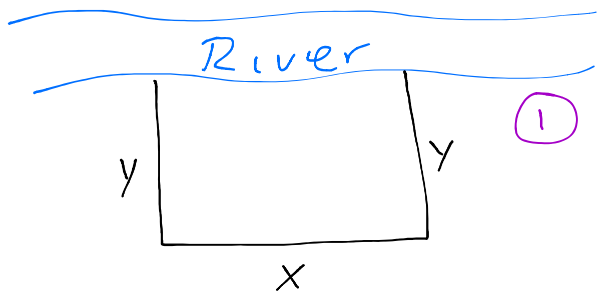
Notes 5.6 – Solving Optimization Problems

General Approach to Solving Optimization Problems

1. Draw a picture and define any variables you may use.
2. Based on what you are supposed to optimize, write your primary equation. If your primary equation is multivariable, make a secondary equation.
3. Use the primary and secondary equations to write the primary equation in one variable.
4. Optimize the primary equation. (Solve $f' = 0$. Determine if max or min by second-derivative test.)
5. Substitute the answer from #4 into the secondary equation to find the value of the other variable.
6. Find the maximum value by substituting into the primary equation.

Example 1

A man lives in a white van down by the river. While spelunking he finds 1000 feet of pristine fence. He decides to build a rectangular enclose along the river to mark his territory. If the side along the river needs no fence, find the dimensions that make his territory as large as possible. Also find the maximum area.



1

x = // to river in feet

y = $\perp$ to river in feet

A : Area

P = Perimeter

2 $A = x \cdot y$

3 $A = (1000 - 2y)\,y$

$A(y) = 1000y - 2y^2$

2 $P = x + y + y$

$1000 = x + 2y$

$1000 - 2y = x$

4 $A'(y) = 1000 - 4y$

$0 = 1000 - 4y$

$4y = 1000$

$y = 250$

$A''(y) = -4$

$A''(y) = (-)$, concave down, MAX

5 $1000 - 2(250) = x$

$1000 - 500 = x$

$500 = x$

6 $A = x\,y$

$= (500)(250)$

$A = 125{,}000 \text{ ft}^2$

In order to maximize the area, the farmer needs to make the side parallel to the river 500 feet and the side perpendicular to the river 250 feet, thus giving him a maximum area of 125,000 square feet.

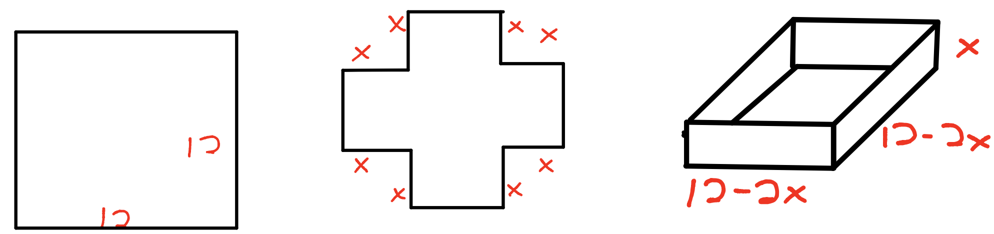
Example 2

An open-top box is to be made from a square sheet of metal 12 inches on each side by cutting a square from each corner and folding up the sides, as in the diagram below. Find the volume of the largest box that can be made.

x = length of cut in inches

V = volume

#1



② $\boxed{\begin{aligned} V &= \text{Area}_{\text{base}} \cdot \text{height} \\ V &= (12-2x)(12-2x)x \\ V &= (144-48x+4x^2)x \\ V &= 4x^3-48x^2+144x \end{aligned}}$

Domain = $(0, 6)$

③ $v'(x) = 12x^2 - 96x + 144$

$0 = 12(x^2 - 8x + 12)$

$0 = 12(x-6)(x-2)$

$0 \neq 12$ $\quad$ $0 = x-6$, $6 = x$ $\quad$ $0 = x-2$, $2 = x$

CV: $x = 2, \cancel{6}$ (Not in Domain)

$v''(x) = 24x - 96$

$v''(x) = 24(x-4)$

$v''(2) = (+)(-) = -$, concave DN

MAX

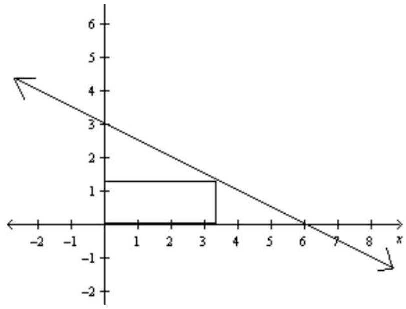
④ CP

$\boxed{V(2) = 128 \text{ in}^3}$

In order to maximize the volume of the box to 128 cubic inches, there needs to be 2 inches cut off the corners giving dimensions of 2" by 8" by 8".

Example 3

A rectangle is bounded by the x and y axes and the graph of $y = 3 - \frac{1}{2}x$. What length and width should the rectangle have so that its area is a maximum?



① $A = x \cdot y$

$A = x(3 - \frac{1}{2}x)$

$A = 3x - \frac{1}{2}x^2$

$A' = 3 - x$

$A' = 0$	$A' = \text{und}$
$0 = 3 - x$	never
$x = 3$	

②

	$+$	$-$
A'	$x=1$	$x=4$

$0 \qquad 3$

Area is a relative max when $x=3$ b/c A' changes from $+$ to $-$ at $x=3$

③

$y = 3 - \frac{1}{2}x$

$y = 3 - \frac{1}{2}(3)$

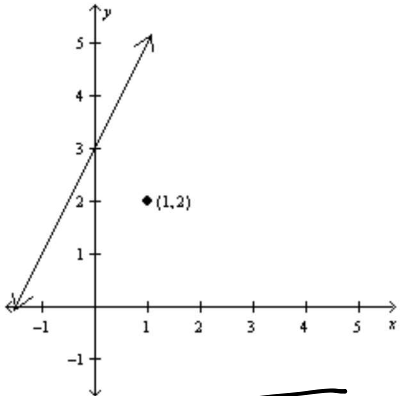
$y = \frac{6}{2} - \frac{3}{2}$

$y = 3/2$

The rectangle will have a maximum area of $9/2$ units if the length, x, equals 3 units and the width, y, is $3/2$ units.

Example 4

Determine the point on the line $y = 2x + 3$ so that the distance between the line and the point (1, 2) is a minimum.



1

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$d = \sqrt{(x-1)^2 + (y-2)^2}$$

$$d = \sqrt{(x-1)^2 + [(2x+3)-2]^2}$$

$$d = \sqrt{x^2 - 2x + 1 + (2x+1)^2}$$

$$d = \sqrt{x^2 - 2x + 1 + 4x^2 + 4x + 1}$$

$$d = (5x^2 + 2x + 2)^{\frac{1}{2}}$$

$$d' = \frac{1}{2}(5x^2 + 2x + 2)^{-\frac{1}{2}}(10x + 2)$$

$$d' = \frac{10x + 2}{\frac{1}{2}(5x^2 + 2x + 2)^{1/2}}$$

$d' = 0$	$d' =$ und
$0 = 10x + 2$	Never
$-2 = 10x$	
$-\frac{1}{5} = x$	

2

d': NEG ($x = -1$) | $-\frac{1}{5}$ | POS ($x = 0$)

d has a relative min at $x = -1/5$ b/c d' changes from negative to positive at $x = -\frac{1}{5}$

3

$$\boxed{\begin{aligned} y &= 2x + 3 \\ y &= 2\left(-\tfrac{1}{5}\right) + 3 \\ y &= -\tfrac{2}{5} + \tfrac{15}{5} \\ y &= \tfrac{13}{5} \end{aligned}}$$

4

The point $\left(-\frac{1}{5}, \frac{13}{5}\right)$ minimizes the distance $y = 2x + 3$ is from $(1, 2)$