Notes 5.7 – L'Hôspital's Rule

L'Hospital's Rule

If $\lim_{x\to c}\frac{f(x)}{g(x)}$ produces the indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

L'Hospital's Rule in action

1. $\lim_{x\to 0}\frac{\sin x}{x}$ $\left\{ \begin{array}{l} \lim_{x\to 0}(\sin x) = 0 \\ \lim_{x\to 0} x = 0 \end{array} \right.$

This limit produces the indeterminate form $\frac{0}{0}$

$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{(\sin x)'}{x'} = \lim_{x\to 0}(\cos x) = 1$

2. $\lim_{x\to 0}\frac{1-\cos(2x)}{5x}$ $\left\{ \begin{array}{l} \lim_{x\to 0}(1-\cos(2x)) = 1-1=0 \\ \lim_{x\to 0}(5x) = 0 \end{array} \right.$

This limit produces the indeterminate form $\frac{0}{0}$

$\lim_{x\to 0}\frac{1-\cos(2x)}{5x} = \lim_{x\to 0}\frac{2\cdot\sin(2x)}{5} = \frac{2\cdot 0}{5} = 0$

Practice 1 - Should we apply L'Hospital's Rule?

a. $\lim_{x\to 0}\frac{\sin x}{2x^2-1}$ $\left\{ \begin{array}{l} \lim_{x\to 0}(\sin x) = 0 \\ \lim_{x\to 0}(2x^2-1) = -1 \end{array} \right.$

L'HOSPITAL'S DOES NOT APPLY

b. $\lim_{x\to 2}\frac{\sin x}{x^2-4}$ $\left\{ \begin{array}{l} \lim_{x\to 2}(\sin x) = \sin(2) \\ \lim_{x\to 2}(x^2-4) = 0 \end{array} \right.$

L'HOSPITAL'S DOES NOT APPLY

c. $\lim_{x\to 2}\frac{\sin(x-2)}{x^2-4}$ $\left\{ \begin{array}{l} \lim_{x\to 2}(\sin(x-2)) = 0 \\ \lim_{x\to 2}(x^2-4) = 0 \end{array} \right.$

L'HOSPITAL'S APPLIES

Practice 2 – Find each limit

1. $\lim_{x\to 1}\frac{\ln x}{x^2-1}$ $\left\{ \begin{array}{l} \lim_{x\to 1}(\ln x) = \ln(1) = 0 \\ \lim_{x\to 1}(x^2-1) = 0 \end{array} \right.$

This limit produces the indeterminate form $\frac{0}{0}$

$\lim_{x\to 1}\frac{\ln(x)}{x^2-1} = \lim_{x\to 1}\frac{(\ln(x))'}{(x^2-1)'} = \lim_{x\to 1}\frac{\frac{1}{x}}{2x} = \frac{\frac{1}{1}}{2} = \frac{1}{2}$

2. The table gives selected values of a twice differentiable function $f(x)$.

x	5	3
$f(x)$	0	-3
$f'(x)$	7	4

Find $\lim_{x\to 3}\frac{f(2x-1)}{x^2-9}$ $\left\{ \begin{array}{l} \lim_{x\to 3} f(2x-1) = f(5) = 0 \\ \lim_{x\to 3}(x^2-9) = 0 \end{array} \right.$

This limit produces the indeterminate form $\frac{0}{0}$

$\lim_{x\to 3}\frac{f(2x-1)}{x^2-9} = \lim_{x\to 3}\frac{(f(2x-1))'}{(x^2-9)'} = \lim_{x\to 3}\frac{2\cdot f'(2x-1)}{2x}$

$= \frac{2\cdot f'(5)}{6}$

$= \frac{2\cdot 7}{6}$

$= \frac{7}{3}$

AP Style Questions

1. $\lim_{x\to 0}\frac{e^x-\cos x-2x}{x^2-2x}$ produces the indeterminate form $\frac{0}{0}$

$= \lim_{x\to 0}\frac{e^x+\sin x-2}{2x-2}$ produces the indeterminate form $\frac{0}{0}$

$= \lim_{x\to 0}\frac{e^x+\cos x}{2} = \frac{1+1}{2} = 1$

   a. -½
   b. 0
   c. ½
   d. 1
   e. Nonexistent

2. $\lim_{x\to\infty}\frac{\overbrace{\ln(e^{3x}+x)}^{\infty}}{\underbrace{x}_{\infty}}$ produces the indeterminate form $\frac{\infty}{\infty}$

$\lim_{x\to\infty}\frac{\frac{3e^{3x}+1}{e^{3x}+x}}{1} = \lim_{x\to\infty}\frac{3e^{3x}+1}{e^{3x}+x}$ indeterminate form $\frac{\infty}{\infty}$

$= \lim_{x\to\infty}\frac{9e^{3x}}{3e^{3x}+1}$ indeterminate form $\frac{\infty}{\infty}$

$= \lim_{x\to\infty}\frac{27\cancel{e^{3x}}}{9\cancel{e^{3x}}}$

$= \lim_{x\to\infty}(3)$

$= 3$

   a. 0
   b. 1
   c. 3
   d. ∞

FRQ 1
The functions *f* and *h* are twice differentiable with $h(2) = 4$.

h is continuous
$\lim_{x\to 2} h(x) = 4$

The function h satisfies $h(x) = \frac{x^2-4}{1-(f(x))^3}, x \neq 2$. It is known that $\lim_{x\to 2} h(x)$ can be evaluated using L'Hospital's Rule.

$\lim_{x\to 2}(x^2-4)=0$

implies

$\lim_{x\to 2}\left[1-(f(x))^3\right] = 0$

Use $\lim_{x\to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

$\lim_{x\to 2}\left[1-(f(x))^3\right] = 0$

f is differentiable ∴ f is continuous

∴ $1-f^3(2) = 0$

$-f^3(2) = -1$

$f^3(2) = 1$

$f(2) = 1$

$\lim_{x\to 2}\frac{x^2-4}{1-(f(x))^3} \overset{\text{L'HOSPITAL'S}}{=} \lim_{x\to 2}\frac{2x}{-3(f(x))^2\cdot f'(x)}$

$4 = \frac{2(2)}{-3(f(2))^2\cdot f'(2)}$

$4 = \frac{4}{-3\cdot(1)^2 f'(2)}$

$4 = \frac{4}{-3\cdot f'(2)}$

$4\cdot\left(\frac{-3}{4}\right) = \frac{1}{f'(2)}$

$-3 = \frac{1}{f'(2)}$

$-3\cdot f'(2) = 1$

$f'(2) = \frac{1}{-3}$

CAUTION: L'Hôpital's Rule is not the Quotient Rule

1. $\frac{d}{dx}\left(\frac{\sin(6x)}{x}\right)$