

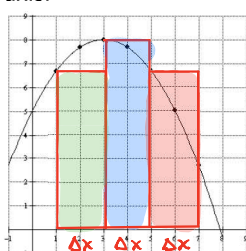
## Notes 6.2 – Riemann Sums

### A Graphical Approach to Approximating the Definite Integral

- In calculus, the result of  $\int f(x)dx$  is a function that represents the anti-derivative of the function  $f(x)$ . This is also sometimes referred to as an INDEFINITE INTEGRAL. The result of  $\int_a^b f(x)dx$  is a value that represents the area of the region bounded by the curve of  $f(x)$  and the  $x$ -axis on the interval  $a \leq x \leq b$ .
- Calculating Riemann sums is a way to **estimate** the area under a curve, the value of  $\int_a^b f(x)dx$ , for a graphed function on a particular interval. In this activity, you will learn to calculate four types of Riemann sums: Left Hand, Right Hand, Midpoint, and Trapezoidal Sums.

#### Approximation #1 – Left Hand Riemann Sum with intervals of length 2 units

On the interval [1, 7] subdivide the area bounded by the function and the  $x$ -axis into rectangles of length 2 unit. axis.



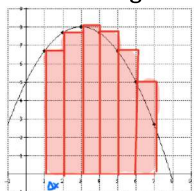
$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = 2$$

$$\begin{aligned} \text{Area} &= \int_1^7 f(x)dx \approx A_1 + A_2 + A_3 \\ &\approx \Delta x \cdot f(1) + \Delta x f(3) + \Delta x f(5) \\ &\approx \Delta x [f(1) + f(3) + f(5)] \\ &\approx 2 [6.667 + 8 + 6.667] \\ \text{Area} &\approx 42.667 \text{ un}^2 \end{aligned}$$

#### Approximation #2 – Left Hand Riemann Sum with intervals of length 1 unit

On the interval [1, 7] subdivide the area bounded by the function and the  $x$ -axis into rectangles of length 1 unit.



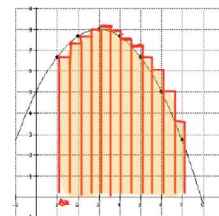
$$\Delta x = 1$$

$$\begin{aligned} \text{Area} &= \int_1^7 f(x)dx = \Delta x [\sum \text{heights}] \\ &\approx 1 \cdot [f(1) + f(2) + f(3) + f(4) + f(5) + f(6)] \end{aligned}$$

$$\text{Area} \approx 41.667 \text{ un}^2$$

#### Approximation #3 – Left Hand Riemann Sum with intervals of length 1/2 unit

On the interval [1, 7] subdivide the area bounded by the function and the  $x$ -axis into rectangles of length 1/2 unit.

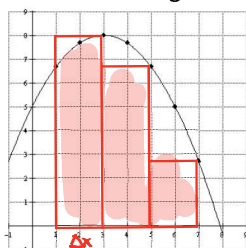


$$\Delta x = \frac{1}{2}$$

$$\begin{aligned} A &= \int_1^7 f(x)dx = \Delta x [\sum \text{heights}] \\ &\approx \frac{1}{2} [f(1) + f(1.5) + f(2) + f(2.5) + f(3) + f(3.5) + f(4) \\ &\quad + f(4.5) + f(5) + f(5.5) + f(6) + f(6.5)] \\ &\approx 40.917 \text{ un}^2 \end{aligned}$$

#### Approximation #4 – Right Hand Riemann Sum with intervals of length 2 units

On the interval [1, 7] subdivide the area bounded by the graph and the  $x$ -axis into rectangles of length 2 units.

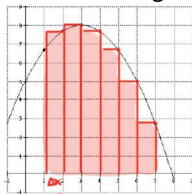


$$\Delta x = 2$$

$$\begin{aligned} A &= \int_1^7 f(x)dx \\ &\approx \Delta x [\sum \text{heights}] \\ &\approx 2 [f(3) + f(5) + f(7)] \\ &\approx 34.667 \text{ un}^2 \end{aligned}$$

#### Approximation #5 – Right Hand Riemann Sum with intervals of length 1 unit

On the interval [1, 7] subdivide the area bounded by the graph and the  $x$ -axis into rectangles of length 1 unit.

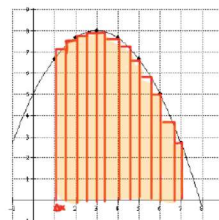


$$\Delta x = 1$$

$$\begin{aligned} A &= \int_1^7 f(x)dx = \Delta x [\sum \text{heights}] \\ &\approx 1 [f(2) + f(3) + f(4) + f(5) + f(6) + f(7)] \\ &\approx 37.667 \text{ un}^2 \end{aligned}$$

#### Approximation #6 – Right Hand Riemann Sum with intervals of length 1/2 unit

On the interval [1, 7] subdivide the area bounded by the function and the  $x$ -axis into rectangles of length 1/2 unit.



$$\Delta x = \frac{1}{2}$$

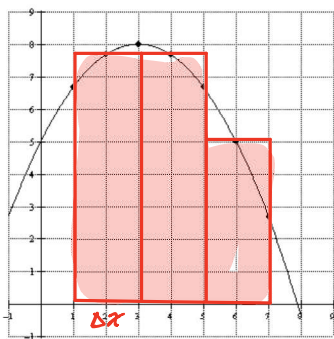
$$\begin{aligned} A &= \int_1^7 f(x)dx \approx \Delta x [\sum \text{heights}] \\ A &\approx \frac{1}{2} [f(1.5) + f(2) + f(2.5) + f(3) + f(3.5) + f(4) + f(4.5) + f(5) \\ &\quad + f(5.5) + f(6) + f(6.5)] \\ A &\approx 38.917 \text{ un}^2 \end{aligned}$$

Before we continue, what inference can you make about the approximations as the lengths of the rectangles decreases?

Graphically, why do you suppose this is so?

**Approximation #7 – Midpoint Riemann Sum with intervals of length 2 units**

On the interval [1, 7] subdivide the area bounded by the function and the x-axis into rectangles of length 2 units.



$\Delta x = 2$

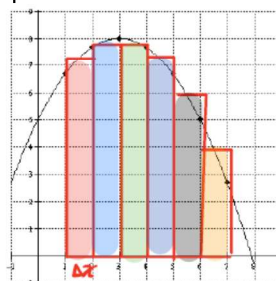
$$A = \int_1^7 f(x) dx \approx \Delta x [\sum \text{heights}]$$

$$\approx 2 [f(2) + f(4) + f(6)]$$

$$\approx 40.667 \text{ un}^2$$

**Approximation #8 – Midpoint Riemann Sum with intervals of length 1 units**

On the interval [1, 7] subdivide the area bounded by the function and the x-axis into rectangles of length 1 unit.



$\Delta x = 1$

$$A = \int_1^7 f(x) dx \approx \Delta x [\sum \text{heights}]$$

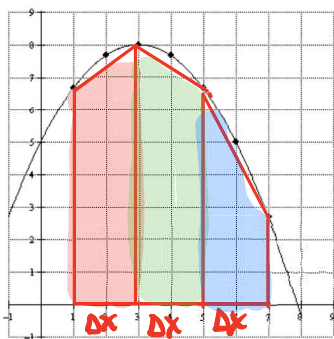
$$\approx 1 [f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5) + f(6.5)]$$

$$\approx 40.167 \text{ un}^2$$

Remember that the area of a trapezoid is found using the formula  $A = \frac{1}{2}h(b_1 + b_2)$ , where  $b_1$  and  $b_2$  are the parallel sides, and  $h$  is the distance between those parallel sides.

**Approximation #9 – Trapezoidal Riemann Sum with intervals of length 2 units**

On the interval [1, 7] subdivide the area bounded by the graph of the function and the x-axis into trapezoids of height 2 units.



$\Delta x = 2$

$$A_{\text{TRAP}} = h \cdot \frac{1}{2}(b_1 + b_2)$$

$$= \Delta x \frac{1}{2} [f(x_1) + f(x_2)]$$

$$A = \int_1^7 f(x) dx \approx A_{\text{TRAP}_1} + A_{\text{TRAP}_2} + A_{\text{TRAP}_3}$$

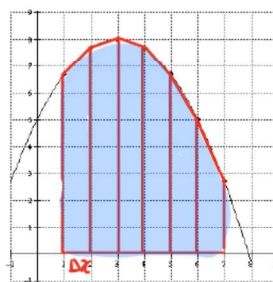
$$\approx \Delta x \frac{1}{2} [f(1) + f(3)] + \Delta x \frac{1}{2} [f(3) + f(5)] + \Delta x \frac{1}{2} [f(5) + f(7)]$$

$$\approx \Delta x \frac{1}{2} [f(1) + 2f(3) + 2f(5) + f(7)]$$

$$\approx 38.667 \text{ un}^2$$

**Approximation #10 – Trapezoidal Riemann Sum with intervals of length 1 unit**

On the interval [1, 7] subdivide the area bounded by the graph of the function and the x-axis into trapezoids of height 1 unit.



$\Delta x = 1$

$$A = \int_1^7 f(x) dx$$

$$\approx \frac{1}{2} (1) [f(1) + 2f(2) + 2f(3) + 2f(4) + 2f(5) + 2f(6) + f(7)]$$

$$\approx 39.667 \text{ un}^2$$

Type of Sum	Numerical Approximation	Do you think this is an over or under approximation of the area?
Left Hand Riemann Sum Intervals of 2 units	42.667	Over
Left Hand Riemann Sum Intervals of 1 unit	41.667	over
Left Hand Riemann Sum Intervals of 1/2 unit	40.917	over
Right Hand Riemann Sum Intervals of 2 units	34.667	under
Right Hand Riemann Sum Intervals of 1 unit	37.667	under
Right Hand Riemann Sum Intervals of 1/2 unit	38.917	under
Midpoint Riemann Sum Intervals of 2 units	40.667	over
★ Midpoint Riemann Sum Intervals of 1 unit	40.167	over ★
Trapezoidal Riemann Sum Intervals of 2 units	38.667	under
Trapezoidal Riemann Sum Intervals of 1 unit	39.667	under

At this point, place a star next to the approximation that you feel is the most accurate to the actual area.

In our next lesson, we will learn how to find the EXACT area of a region between the graph of a function and the x – axis.

In the space below, we will come back to this to find the exact area once we complete the next lesson in order to see which approximation to the left is the most accurate.

$$\begin{aligned}
 \int_1^7 \left(-\frac{1}{3}x^2 + 2x + 5\right) dx &= \left[-\frac{1}{9}x^3 + x^2 + 5x\right]_1^7 \\
 &= \left[-\frac{1}{9}(7)^3 + (7)^2 + 5(7)\right] - \left[-\frac{1}{9}(1)^3 + (1)^2 + 5(1)\right] \\
 &= \left[-\frac{343}{9} + 49 + 35\right] - \left[-\frac{1}{9} + 1 + 5\right] \\
 &= -\frac{343}{9} + 84 + \frac{1}{9} - 6 \\
 &= -\frac{342}{9} + 78 \\
 &= -38 + 70 \\
 &= 40
 \end{aligned}$$

Given the table of values below, approximate each definite integral by finding the indicated Riemann Sum.

a. Approximate  $\int_0^{25} f(x)dx$  using a midpoint sum and three subintervals.

x	0	4	7	12	15	20	25
f(x)	15	6	-5	-10	-2	8	20

$$\int_0^{25} f(x)dx \approx 7(6) + 8(-10) + 10(8) = 42$$

b. Approximate  $\int_0^{15} f(x)dx$  using a left-hand sum and four subintervals.

x	0	4	7	12	15	20	25
f(x)	15	6	-5	-10	-2	8	20

$$\int_0^{15} f(x)dx \approx 4(15) + 3(6) + 5(-5) + 3(-10) = 23$$

c. Approximate  $\int_4^{20} f(x)dx$  using a right-hand sum and four subintervals.

x	0	4	7	12	15	20	25
f(x)	15	6	-5	-10	-2	8	20

$$\int_4^{20} f(x)dx \approx 3(-5) + 5(-10) + 3(-2) + 5(8) = -31$$

d. Approximate  $\int_0^{25} f(x)dx$  using a trapezoidal sum and three subintervals.

x	0	4	7	12	15	20	25
f(x)	15	6	-5	-10	-2	8	20

$$\int_0^{25} f(x)dx \approx \frac{1}{2}(7)[15 + -5] + \frac{1}{2}(8)[-5 + -2] + \frac{1}{2}(10)[-2 + 20] = 97$$