Notes 7.3 – Solving Differential Equations

Examples of Variable Separable Differential Equations

Given below are differential equations with given initial condition values. Find the particular solution, $y = f(x)$, for each differential equation that satisfies the given initial condition.

1. $\frac{dy}{dx} = 6x^2 + 6x + 2$ and $f(-1) = 2$

$dy = (6x^2+6x+2)\,dx$

$\int dy = \int (6x^2+6x+2)\,dx$

$y = 2x^3 + 3x^2 + 2x + C$ (General Solution)

@ $(-1,2)$

$2 = 2(-1)^3 + 3(-1)^2 + 2(-1) + C$

$2 = -2 + 3 - 2 + C$

$3 = C$

$f(x) = 2x^3 + 3x^2 + 2x + C$ (specific solution)

2. $\frac{dy}{dx} = \frac{1+12x^{3/2}}{2\sqrt{x}}$ and $f(0) = 2$

$dy = (1+12x^{3/2})\frac{1}{2}x^{-1/2}\,dx$

$dy = \left(\frac{1}{2}x^{-1/2} + 6x^{2/2}\right)dx$

$\int dy = \int \left(\frac{1}{2}x^{-1/2} + 6x\right)dx$

$y = x^{\frac{1}{2}} + 3x^2 + C$ (General Solution)

at $(0,2)$

$2 = \sqrt{0} + 3(0)^2 + C$

$2 = C$

$f(x) = \sqrt{x} + 3x^2 + 2$ (Specific solution)

3. $\frac{dy}{dx} = \frac{x^2+2x}{2y}$ and $f(0) = 2$

$2y\,dy = (x^2+2x)\,dx$

$\int 2y\,dy = \int (x^2+2x)\,dx$

$y^2 = \frac{1}{3}x^3 + x^2 + C$

at $(0,2)$

$2^2 = \frac{1}{3}(0)^3 + (0)^2 + C$

$4 = C$

$y^2 = \frac{1}{3}x^3 + x^2 + 4$

$y = \pm\sqrt{\frac{1}{3}x^3 + x^2 + 4}$

which one contains $(0,2)$?

$f(x) = \sqrt{\frac{1}{3}x^3 + x^2 + 4}$ (specific solution)

4. $\frac{dy}{dx} = \frac{x+2}{y}$ and $f(1) = -3$

$y\,dy = (x+2)\,dx$

$\int y\,dy = \int (x+2)\,dx$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + 2x + C$

$y^2 = x^2 + 4x + C$

at $(1,-3)$

$(-3)^2 = (1)^2 + 4(1) + C$

$9 = 1 + 4 + C$

$9 = 5 + C$

$4 = C$

$y = \pm\sqrt{x^2 + 4x + 4}$

which one contains $(1,-3)$?

$f(x) = -\sqrt{x^2 + 4x + 4}$ (specific solution)

5. $\frac{dy}{dx} = x^4(y-2)$ and $f(0) = 0$

$$dy = x^4(y-2)\,dx$$

$$\int \frac{1}{y-2}\,dy = \int x^4\,dx$$

$u = y-2$

$\frac{du}{dy} = 1$

$du = dy$

$$\int \frac{1}{u}\,du = \frac{1}{5}x^5 + C$$

$$\ln|u| = \frac{1}{5}x^5 + C$$

$$\ln|y-2| = \frac{1}{5}x^5 + C$$

at $(0,0)$

$$\ln|0-2| = \frac{1}{5}(0)^5 + C$$

$$\ln|-2| = C$$

$$\ln 2 = C$$

$$\ln|y-2| = \frac{1}{5}x^5 + \ln 2$$

$$|y-2| = e^{\frac{1}{5}x^5 + \ln 2}$$

$$= e^{\frac{1}{5}x^5} \cdot e^{\ln 2}$$

$$|y-2| = 2e^{\frac{1}{5}x^5}$$

$y - 2 = -2e^{\frac{1}{5}x^5}$ $\qquad$ $y - 2 = 2e^{\frac{1}{5}x^5}$

$y = -2e^{\frac{1}{5}x^5} + 2$ $\qquad$ $y = 2e^{\frac{1}{5}x^5} + 2$

which one contains $(0,0)$?

$0 = -2e^{\frac{1}{5}(0)^5} + 2$ $\qquad$ $0 = 2e^{\frac{1}{5}(0)} + 2$

$0 = -2(1) + 2$ $\qquad$ $0 = 2(1) + 2$

$0 = 0$ $\qquad$ $0 \neq 4$

$\therefore f(x) = -2e^{\frac{1}{5}x^5} + 2$

6. $\frac{dy}{dx} = \frac{y-1}{x^2}$ and $f(2) = 0$

$$x^2\,dy = (y-1)\,dx$$

$u = y-1$

$du = dy$

$$\int \frac{1}{y-1}\,dy = \int x^{-2}\,dx$$

$$\int \frac{1}{u}\,du = -x^{-1} + C$$

$$\ln|u| = -\frac{1}{x} + C$$

$$\ln|y-1| = -\frac{1}{x} + C$$

at $(2,0)$

$$\ln|0-1| = -\frac{1}{2} + C$$

$$\ln|-1| = -\frac{1}{2} + C$$

$$\ln 1 = -\frac{1}{2} + C$$

$$0 = -\frac{1}{2} + C$$

$$\frac{1}{2} = C$$

$$\ln|y-1| = -\frac{1}{x} + \frac{1}{2}$$

$$|y-1| = e^{-\frac{1}{x} + \frac{1}{2}}$$

$$|y-1| = e^{-\frac{1}{x} + \frac{1}{2}}$$

$y - 1 = e^{-\frac{1}{x} + \frac{1}{2}}$ or $y - 1 = -e^{-\frac{1}{x} + \frac{1}{2}}$

$y = e^{-\frac{1}{x} + \frac{1}{2}} + 1$ $\qquad$ $y = -e^{-\frac{1}{x} + \frac{1}{2}} + 1$

which one contains $(2,0)$?

$y = e^{-\frac{1}{x} + \frac{1}{2}} + 1$ $\qquad$ $f(x) = -e^{-\frac{1}{x} + \frac{1}{2}} + 1$

NO $\qquad$ Yes

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Question 6

Consider the differential equation $\frac{dy}{dx} = \frac{3x^2}{e^{2y}}$.

(a) Find a solution $y = f(x)$ to the differential equation satisfying $f(0) = \frac{1}{2}$.

$$\frac{dy}{dx} = \frac{3x^2}{e^{2y}}$$

$$\int e^{2y}\,dy = \int 3x^2\,dx$$

$$\frac{1}{2}e^{2y} = x^3 + C$$

at $(0, \frac{1}{2})$

$$\frac{1}{2}e^{2(\frac{1}{2})} = (0)^3 + C$$

$$\frac{1}{2}e = C$$

$$\frac{1}{2}e^{2y} = x^3 + \frac{1}{2}e$$

$$e^{2y} = 2x^3 + e$$

$$2y = \ln(2x^3 + e)$$

$$y = \frac{1}{2}\ln(2x^3 + e)$$

$$f(x) = \frac{1}{2}\ln(2x^3 + e)$$

(b) Find the domain and range of the function f found in part (a).

Domain $\left(\sqrt[3]{-\frac{e}{2}}, \infty\right)$ Range $(-\infty, \infty)$

Arguement > 0

$$2x^3 + e > 0$$

$$2x^3 > -e$$

$$x^3 > -e/2$$

$$x > \sqrt[3]{\frac{-e}{2}}$$

Not part of answer

$y = \log_a x$