

Notes 1.5 – Limits of Trigonometric Functions*An Analytical Approach*

We have already looked at how to evaluate limits of trigonometric functions by direct substitution, provided that the function is defined and continuous at θ . Find the limit.

$$\lim_{\theta \rightarrow \pi} 2 \cos^2 \theta = 2 \left[\lim_{\theta \rightarrow \pi} \cos \theta \right]^2 = 2 [\cos(\pi)]^2 = 2 (-1)^2 = 2 \cdot 1 = 2$$

Not all functions are undefined at a value, but their limits do exist. The same is true in the trigonometric world.

Evaluating Trigonometric Limits by Rewriting the Function Using Identities

$$1. \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin^2 \theta} = \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{1 - \cos^2 \theta}$$

$$\begin{aligned} \sin^2 \theta &+ \cos^2 \theta = 1 \\ \sin^2 \theta &= 1 - \cos^2 \theta \end{aligned}$$

$$\begin{aligned} &= \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{(1 - \cos \theta)(1 + \cos \theta)} = \lim_{\theta \rightarrow 0} \frac{1}{1 + \cos \theta} \\ &= \frac{1}{1 + \cos(0)} = \frac{1}{1 + 1} = \frac{1}{2} \end{aligned}$$

$$2. \lim_{\theta \rightarrow \frac{3\pi}{2}} (3 \tan \theta \cos \theta) = \lim_{\theta \rightarrow \frac{3\pi}{2}} (3 \cdot \frac{\sin \theta}{\cos \theta} \cos \theta)$$

$$= \lim_{\theta \rightarrow \frac{3\pi}{2}} (3 \sin \theta) = 3 \cdot \sin\left(\frac{3\pi}{2}\right)$$

$$= 3(-1)$$

$$= -3$$

$$3. \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\sec \theta \cos \theta}{4\theta} = \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{1}{4\theta}$$

$$= \frac{1}{4\left(\frac{\pi}{2}\right)}$$

$$= \frac{1}{2\pi}$$

$$4. \lim_{\theta \rightarrow \pi} \frac{\cos \theta \tan \theta}{\sin \theta} = \lim_{\theta \rightarrow \pi} \frac{\cos \theta \cdot \frac{\sin \theta}{\cos \theta}}{\sin \theta}$$

$$= \lim_{\theta \rightarrow \pi} \frac{\sin \theta}{\sin \theta} = \lim_{\theta \rightarrow \pi} 1$$

$$= 1$$

Special Trig Limits

$$\lim_{x \rightarrow 0} \frac{\sin(cx)}{cx} = 1$$

$$\lim_{x \rightarrow 0} \frac{cx}{\sin(cx)} = 1$$

$$\lim_{x \rightarrow 0} \frac{\cos(cx) - 1}{cx} = 0$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos(cx)}{cx} = 0$$

where c is any constant

Find each of the following limits.

$$1. \lim_{x \rightarrow 0} \frac{e^x \cos x}{4} = \frac{e^0 \cos(0)}{4}$$

$$= \frac{1 \cdot 1}{4}$$

$$= \frac{1}{4}$$

$$2. \lim_{\theta \rightarrow 0} \frac{\sin(4\theta)}{\theta} = 4 \lim_{\theta \rightarrow 0} \frac{\sin(4\theta)}{4\theta}$$

$$\begin{aligned} \text{spec} \\ \lim_{x \rightarrow 0} \frac{\sin(cx)}{cx} &= 1 \end{aligned}$$

$$3. \lim_{x \rightarrow 0} \frac{\sin(2x)}{3x} = \frac{1}{3} \cdot 2 \lim_{x \rightarrow 0} \frac{\sin(2x)}{2x}$$

$$= \frac{2}{3} \cdot 1$$

$$= \frac{2}{3}$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos(cx)}{cx} = 0$$

$$4. \lim_{\theta \rightarrow 0} \frac{2 \sin 5\theta}{3\theta} = \frac{2}{3} \lim_{\theta \rightarrow 0} \frac{5 \cdot \sin(5\theta)}{5\theta}$$

$$= \frac{2}{3} \cdot 5 \cdot 1$$

$$= \frac{10}{3}$$

$$5. \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = \lim_{\theta \rightarrow 0} \left(\tan \theta \cdot \frac{1}{\theta} \right)$$

$$= \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\cos \theta} \cdot \frac{1}{\theta} \right)$$

$$= \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{1}{\cos \theta} \right)$$

$$= 1 \cdot \frac{1}{\cos(0)}$$

$$= 1 \cdot 1$$

$$= 1$$

$$6. \lim_{\theta \rightarrow 0} \frac{2 - 2 \cos^2 \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{2(1 - \cos^2 \theta)}{\theta}$$

$$= 2 \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta}$$

$$= 2 \cdot \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \cdot \sin \theta \right)$$

$$= 2(1) \cdot \sin(0)$$

$$= 2 \cdot 1 \cdot 0$$

$$= 0$$

$$(3x^2 - 4x - 9)$$

$$7. \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta + \sin 2\theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} + 2 \lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{2\theta}$$

$$= 0 + 2 \cdot (1)$$

$$= 2$$

$$8. \lim_{\theta \rightarrow 0} \frac{\theta \csc \theta + 1}{\theta \csc \theta} = \lim_{\theta \rightarrow 0} \left(\frac{\cancel{\theta \csc \theta}}{\cancel{\theta \csc \theta}} + \frac{1}{\theta \csc \theta} \right)$$

$$= \lim_{\theta \rightarrow 0} \left(1 + \frac{\sin \theta}{\theta} \right)$$

$$= 1 + 1$$

$$= 2$$

$$9. \lim_{x \rightarrow 0} \frac{\sin x - \sin x \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{\sin x (1 - \cos x)}{x^2}$$

$$= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \cdot \frac{1 - \cos x}{x} \right)$$

$$= 1 \cdot 0$$

$$= 0$$