Notes 1.8 – Infinite Limits and Limits at Infinity

Infinite Limits

An infinite limit is a limit that results in $\pm\infty$ when $x \to a$, where a is a real number.

Justification of the Existence of a Vertical Asymptote Using Limits

$x = a$ is a vertical asymptote of $f(x)$ iff $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$

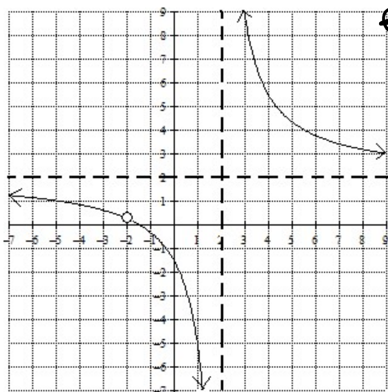
1. $f(x) = \dfrac{2x^2+7x+6}{x^2-4}$

$\lim_{x\to 2^-} f(x) = -\infty$

$\lim_{x\to 2^+} f(x) = \infty$

$\lim_{x\to -1^-} g(x) = -\infty$

$\lim_{x\to -1^+} g(x) = \infty$



Find the limit indicated. Explain what the result of the limit means about the graph of the given rational function.

2. $\lim_{x\to 2^-} \frac{2x^2+7x+6}{x^2-4} = \lim_{x\to 2^-} \frac{(2x+3)\cancel{(x+2)}}{\cancel{(x+2)}(x-2)}$

$= \lim_{x\to 2^-} \frac{2x+3}{x-2}$

$= -\infty$

x	$\frac{2x+3}{x-2}$
1.9	$\frac{+}{-} = -$

$\therefore$ f(x) has VA @ $x = 2$

3. $\lim_{x\to 2^+} \frac{2x^2+7x+6}{x^2-4} = \lim_{x\to 2^+} \frac{(2x+3)\cancel{(x+2)}}{\cancel{(x+2)}(x-2)}$

$= \lim_{x\to 2^+} \frac{2x+3}{x-2}$

$= \infty$

x	$\frac{2x+3}{x-2}$
2.1	$\frac{+}{+} = +$

$\therefore$ f(x) has VA @ $x = 2$

An infinite limit always yields a vertical asymptote.

Limits at Infinity

A limit at infinity is a limit in which $x \to \infty$ or $x \to -\infty$

Justification of the Existence of a Horizontal Asymptote Using Limits

$y = c$ is a horizontal asymptote of $f(x)$ iff $\lim_{x\to -\infty} f(x) = c$ or $\lim_{x\to\infty} f(x) = c$

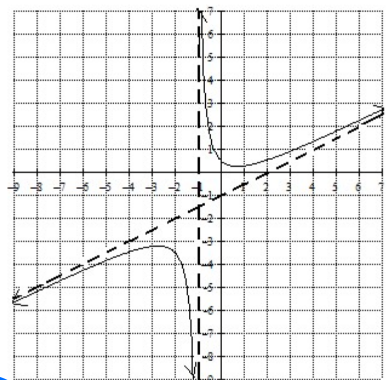
4. $g(x) = \dfrac{x^2-x+1}{2x+2}$

$\lim_{x\to\infty} f(x) = 2$

$\lim_{x\to -\infty} f(x) = 2$

$\lim_{x\to\infty} g(x) = \infty$

$\lim_{x\to -\infty} g(x) = -\infty$



Find each of the following limits at infinity. Explain how you arrived at your answer.

5. $\lim_{x\to\infty} (-3x^3 + x - 4) = -\infty$

neg ODD

Know
1) function even/odd degree
2) function leading term +/-

The function is odd degree with lead coeffient negative. $\therefore$ the graph falls to the right

6. $\lim_{x\to -\infty} (4-x)^2 (x-3)(x+1) = \infty$

$+x^4 \ldots$

Even degree
+ lead coef.

The function is even degree and lead coefficient is +
$\therefore$ graph rises to left

7. $\lim_{x\to\infty} \frac{3-2x}{x+3} = -2$

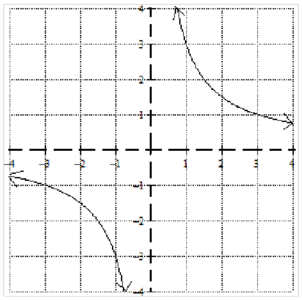
$-2x+3$

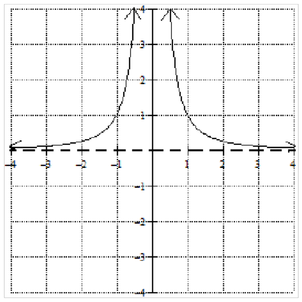
Numerator & denominator grow at same rate
$\therefore$ HA at $y = \frac{a}{b} = \frac{-2}{1}$
$\therefore$ graph approaches -2 as $x \to \infty$

Graphically a limit at infinity can yield a horizontal or slant asymptote. or $\pm\infty$

The third example, $\lim_{x\to\infty} \frac{3-2x}{x+3}$ will provide us a basis for developing our analytical process by which we can find limits at infinity for all types of rational functions. Before we do that, investigate the two functions below both graphically and numerically.

$f(x) = \frac{3}{x}$ $\qquad\qquad$ $g(x) = \frac{1}{x^2}$





What does each of these functions have in common algebraically and what do they have in common graphically?

① Algebraically, the denominator grows faster than numerator.

② Both graphs have HA at $y=0$

HA at $y=0$ if denominator grows faster than numerator

HA at $y = \frac{a}{b}$ if denominator & numerator grow at same rate.

NO HA if numerator grows faster than denom.

In pre-calculus, we learned three rules for determining the existence of horizontal asymptotes of rational functions. When a rational function had a horizontal asymptote, the end behavior was always such that as $x \to \infty$ or $x \to -\infty$, then the graph of $f(x) \to$ the horizontal asymptote. We learned three rules for determining the horizontal asymptote, if one existed, for rational functions. We are about to use the idea of a limit and calculus to find out why those rules are such as they are.

$\lim_{x\to\infty} \frac{1}{x} = 0$

x	1	10	100	1000
$\frac{1}{x}$	1	.1	.01	.001

8. For each function below, divide every term in both the numerator and the denominator by the highest power of x that appears in the denominator. Then, evaluate the indicated limit. Does the result of each limit make sense based on the graph that is pictured?

$$\lim_{x\to\infty} \frac{x+2}{x^2+5x+6} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

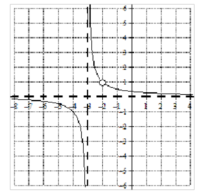
$$= \lim_{x\to\infty} \frac{\frac{x}{x^2} + \frac{2}{x^2}}{\frac{x^2}{x^2} + \frac{5x}{x^2} + \frac{6}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x} + \frac{2}{x^2}}{1 + \frac{5}{x} + \frac{6}{x^2}}$$

$$= \frac{0+0}{1+0+0}$$

$$= \frac{0}{1}$$

$$= 0$$



$$\lim_{x\to-\infty} \frac{x+2}{x^2+5x+6}$$

$$= \lim_{x\to-\infty} \frac{\frac{x}{x^2} + \frac{2}{x^2}}{\frac{x^2}{x^2} + \frac{5x}{x^2} + \frac{6}{x^2}}$$

$$= \lim_{x\to-\infty} \frac{\frac{1}{x} + \frac{2}{x^2}}{1 + \frac{5}{x} + \frac{6}{x^2}}$$

$$= \frac{0+0}{1+0+0}$$

$$= \frac{0}{1}$$

$$= 0$$

$$\lim_{x\to\infty} \frac{2x^2+7x+6}{x^2+5x+6}$$

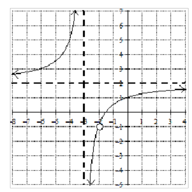
$$= \lim_{x\to\infty} \frac{\frac{2x^2}{x^2} + \frac{7x}{x^2} + \frac{6}{x^2}}{\frac{x^2}{x^2} + \frac{5x}{x^2} + \frac{6}{x^2}}$$

$$= \lim_{x\to\infty} \frac{2 + \frac{7}{x} + \frac{6}{x^2}}{1 + \frac{5}{x} + \frac{6}{x^2}}$$

$$= \frac{2+0+0}{1+0+0}$$

$$= 2/1$$

$$= 2$$



$$\lim_{x\to-\infty} \frac{2x^2+7x+6}{x^2+5x+6}$$

$$= \lim_{x\to-\infty} \frac{2 + \frac{7}{x} + \frac{6}{x^2}}{1 + \frac{5}{x} + \frac{6}{x^2}}$$

$$= \frac{2+0+0}{1+0+0}$$

$$= \frac{2}{1}$$

$$= 2$$

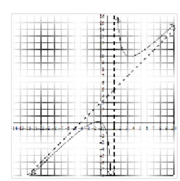
$$\lim_{x\to\infty} \frac{x^2+3x+2}{x-1}$$

$$= \lim_{x\to\infty} \frac{\frac{x^2}{x} + \frac{3x}{x} + \frac{2}{x}}{\frac{x}{x} - \frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{x + 3 + \frac{2}{x}}{1 - \frac{1}{x}}$$

$$= \frac{\infty + 3 + 0}{1 - 0}$$

$$= \infty, \text{dne}$$



$$\lim_{x\to-\infty} \frac{x^2+3x+2}{x-1}$$

$$= \lim_{x\to-\infty} \frac{x + 3 + \frac{2}{x}}{1 - \frac{1}{x}}$$

$$= \frac{-\infty + 3 + 0}{1 - 0}$$

$$= -\infty, \text{dne}$$

Suppose the following limits were given to you in multiple choice. What would the result of the limit be? Base your reasoning on your intuitive understanding of the graph of the function.

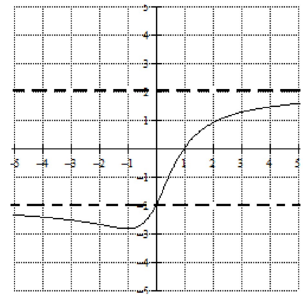
$$\lim_{x\to-\infty}\frac{5-2x-2x^2}{3x^2+2x-3} = -\frac{2}{3}$$

HA

$$\lim_{x\to-\infty}\frac{2x^2-x+5}{3-x} = \infty$$

NO HA

The algebraic analysis described above to evaluate a limit at infinity can be used to find limits at infinity for any type of rational function, even $f(x) = \frac{2x-2}{\sqrt{x^2+1}}$, whose graph is pictured to the right. What is the one thing that you notice is different about the graph of this rational function versus the others that we have investigated in the past?



9. Use the graph to find each of the following limits.

$\lim_{x\to-\infty}\frac{2x-2}{\sqrt{x^2+1}} = -2$ $\lim_{x\to\infty}\frac{2x-2}{\sqrt{x^2+1}} = 2$

10. $\lim_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}} =$

If $x<0$, then $\sqrt{x^2} = -x$

$$\lim_{x\to-\infty} \frac{\frac{3x}{-x} - \frac{2}{-x}}{\sqrt{\frac{2x^2}{x^2} + \frac{1}{x^2}}}$$

$$= \lim_{x\to-\infty} \frac{-3 + \frac{2}{x}}{\sqrt{2+\frac{1}{x^2}}}$$

$$= \frac{-3+0}{\sqrt{2+0}}$$

$$= \frac{-3}{\sqrt{2}}$$

11. $\lim_{x\to-\infty} \frac{x^2+2x}{\sqrt{3x^2+2}} =$

If $x<0$, then $\sqrt{x^2} = -x$

$$\lim_{x\to-\infty} \frac{\frac{x^2}{-x} + \frac{2x}{-x}}{\sqrt{\frac{3x^2}{x^2} + \frac{2}{x^2}}}$$

$$= \lim_{x\to-\infty} \frac{-x-2}{\sqrt{3+\frac{2}{x^2}}}$$

$$= \frac{-(-\infty)-2}{\sqrt{3+0}}$$

$= \infty$, dne