Unit 5 Progress Check: FRQ Part A

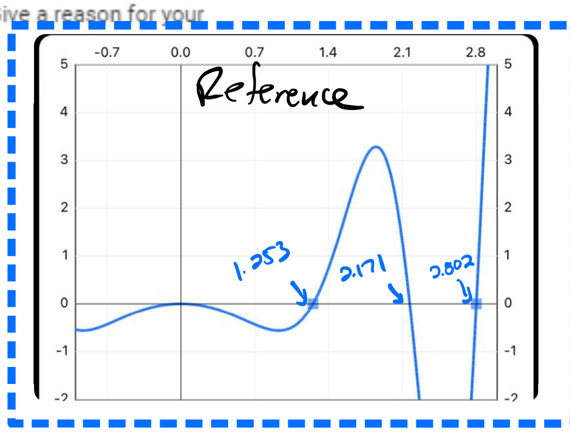
Question 1

Let f be a twice-differentiable function such that $f'(1) = 0$. The second derivative of f is given by $f''(x) = x^2 \cos(x^2 + \pi)$ for $-1 \le x \le 3$.

(a) On what open intervals contained in $-1 < x < 3$ is the graph of f concave up? Give a reason for your answer.

$f'' > 0$ on $(1.253, 2.171)$ and $(2.802, 3)$ +1

$\therefore$ f is concave up there +1



(b) Does f have a relative minimum, a relative maximum, or neither at $x = 1$? Justify your answer.

$f'(1) = 0$ and $f''(1) = -0.540$
$\therefore f''(1) < 0$ +1

$\therefore$ By 2nd Derivative Test, f has a relative maximum at $x = 1$ +1

(c) Use the Mean Value Theorem on the closed interval $[-1, 1]$ to show that $f'(-1)$ cannot equal 2.5.

3 points
TOO HARD

Proof by Contradiction

If $f'(-1) = 2.5$, then $\dfrac{f'(-1) - f'(1)}{-1-1} = \dfrac{2.5 - 0}{-2} = \dfrac{2.5}{-2} = -1.25$

Since f' is continuous on $[-1,1]$ and differentiable on $(-1,1)$, then by the MVT there must be a number c on $(-1,1)$ such that $f''(c) = -1.25$.

However, $-1 \le f''(x) = x^2\cos(x^2+\pi) \le 1$ on $(-1,1)$

$\therefore$ there is no c in $(-1,1)$ such that $f''(c) = -1.25$

$\therefore f'(-1) \ne 2.5$

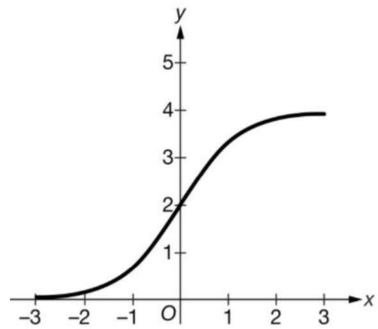
(d) Does the graph of f have a point of inflection at $x = 0$? Give a reason for your answer.

$f''(x) = 0$ at $x = 0$, but $f''(x)$ does not change sign at $x = 0$. +1

$\therefore f(x)$ does not have a point of inflection there. +1

Unit 5 Progress Check: FRQ Part B

Question 1



Graph of g

The graph of the continuous function g is shown above for $-3 \le x \le 3$. The function g is twice differentiable, except at $x = 0$.

Let f be the function with $f(0) = -1$ and derivative given by $f'(x) = \left(x^2 - \frac{5}{4}\right) e^x$.

(a) Find the x-coordinate of each critical point of f. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

$f'(x) = 0$ at $x = -\frac{\sqrt{5}}{2}$ and $x = \frac{\sqrt{5}}{2}$ +1

f': POS | $-\frac{\sqrt{5}}{2}$ | NEG | $\frac{\sqrt{5}}{2}$ | POS

f' changes from positive to negative at $x = -\frac{\sqrt{5}}{2}$ ∴ f has a relative maximum there +1

f' changes from negative to positive at $x = \frac{\sqrt{5}}{2}$ ∴ f has a relative minimum there. +1

(b) Find all values of x at which the graph of f has a point of inflection. Give reasons for your answers.

+1 $f'' = (2x)e^x + \left(x^2 - \frac{5}{4}\right)e^x$ (Product Rule)

$f'' = e^x\left(x^2 + 2x - \frac{5}{4}\right)$ (GCF of e^x)

f'': POS | $-5/2$ | NEG | $1/2$ | POS

$\boxed{f'' = 0}$

$x = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(-5/4)}}{2(1)}$

$x = \dfrac{-2 \pm \sqrt{4+5}}{2}$

$x = \dfrac{-2 \pm 3}{2} = -\frac{5}{2}, \frac{1}{2}$

$f'' = 0$ at $x = -5/2$ and $x = \frac{1}{2}$

f'' changes signs there +1

∴ f has point of inflection there.

(c) Fill in the missing entries in the table below to describe the behavior of g' and g'' on the interval $-3 \le x \le 3$. Indicate Positive or Negative. Give reasons for your answers.

x	$x = -3$	$-3 < x < 0$	$x = 0$	$0 < x < 3$	$x = 3$
$g(x)$	0	Positive	2	Positive	4
$g'(x)$	0	Positive	2	Positive	0
$g''(x)$	0	Positive	Undefined	Negative	0

g is increasing on $(-3,0)$ and $(0,3)$ ∴ g' is positive there

g is concave up on $(-3,0)$ ∴ g'' is positive there

g is concave down on $(0,3)$ ∴ g'' is negative there. +2

(d) Let h be the function defined by $h(x) = f(x)\,g(x)$. Is h increasing or decreasing at $x = 0$? Give a reason for your answer.

$h'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$ +1

$h'(0) = f'(0)\cdot g(0) + f(0)\cdot g'(0)$

$= (\text{NEG})\cdot(\text{POS}) + (-1)(\text{POS})$

$= \text{NEG} + \text{NEG}$

$h'(0) = \text{NEG}$

$h'(x) < 0$ at $x = 0$

∴ h is decreasing there. +1

Unit 5 Progress Check: FRQ Part B

Question 2

The number of mosquitoes in a field after a major rainfall is modeled by the function M defined by $M(t) = -t^3 + 12t^2 + 144t$, where t is the number of days after the rainfall ended and $0 \le t \le 18$.

(a) Using correct units, interpret the meaning of $M'(15) = -171$ in the context of the problem.

15 days after rainfall has ended, the number of mosquitoes in a field is decreasing by 171 mosquitoes per day. +1

$144 \times 12 = 288 + 1440 = 1728$

(b) Based on the model, what is the absolute maximum number of mosquitoes in the field over the time interval $0 \le t \le 18$? Justify your answer.

$m'(t) = -3t^2 + 24t + 144$ +1

$m'(t) = 0$ +1

$-3(t^2 - 8t - 48) = 0$

$-3(t-12)(t+4) = 0$

$\cancel{t = -4}$, $t = 12$ +1

t	$M(t)$
0	0
12	1728 ☆
18	216

$m(12) = -(12)^3 + 12(12)^2 + 144(12)$

$= 1728$

$m(18) = -(18)^3 + 12(18)^2 + 144(18)$

$= -5832 + 3888 + 2160$

$= 216$

By CANDIDATE'S TEST, The absolute maximum number is 1728 +1

(c) For what values of t is the rate of change of the number of mosquitoes in the field increasing?

$m''(t) = -6t + 24$ +1

$m'' = 0$ at $t = 4$

$m''(t)$: POS on $(0, 4)$, NEG on $(4, 18)$

The rate of change of # mosquitoes in the field is increasing on $(0,4)$ +1

(d) For $0 \le t \le 18$, the number of bats in the field is modeled by the differentiable function B, where B is a function of the number of mosquitoes in the field. Based on the models, write an expression for the rate of change of the number of bats in the field at time $t = a$.

$B(M(a))$ = # of bats in field

$B'(M(a)) \cdot M'(a) = B'(-a^3 + 12a^2 + 144a) \cdot (-3a^2 + 24a + 144)$ +1