

APC - Advanced Techniques

6.2A – Related Rates

Hint: the volume of a sphere exists ☺

Chocolate

#1) George left his large ball of chocolate in the sun so that its radius is decreasing at the rate of 2 inches per minute. How fast is the volume decreasing at the moment when the radius is 3 inches?

$$\frac{dr}{dt} = \frac{-2 \text{ in}}{1 \text{ min}}$$

$$\text{FIND } \frac{dV}{dt} \Big|_{r=3}$$

$r = \text{radius (inches)}$
 $t = \text{time (minutes)}$
 $V = \text{Volume (in}^3\text{)}$

$$V = \frac{4}{3} \pi r^3$$

$$\frac{d(V)}{dt} = \frac{d}{dt} \left(\frac{4}{3} \pi r^3 \right)$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 (-2)$$

$$\frac{dV}{dt} = -8\pi r^2$$

$$\frac{dV}{dt} \Big|_{r=3} = -8\pi (3)^2$$

$$= -8\pi (9)$$

$$= -72\pi$$

$$\frac{dV}{dt} \Big|_{r=3} \approx -226.2 \text{ in}^3/\text{min}$$

Sentence Answer:

When the radius of the ball of choc is 3 inches, the volume is decreasing by 226.2 in³ per minute.

Gallstones

#2) A gallstone is forming in George's gallbladder so that its radius is growing at the rate of 1 centimeter per year. How fast is its volume growing at the moment when the radius is 2 centimeters?

$$\frac{dr}{dt} = \frac{1 \text{ cm}}{1 \text{ yr}}$$

$$\text{FIND } \frac{dV}{dt} \Big|_{r=2}$$

$r = \text{radius (cm)}$
 $t = \text{time (year)}$
 $V = \text{Volume (cm}^3\text{)}$

$$V = \frac{4}{3} \pi r^3$$

$$\frac{d}{dt} V = \frac{d}{dt} \left(\frac{4}{3} \pi r^3 \right)$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 (1)$$

$$\frac{dV}{dt} = 4\pi r^2$$

$$\frac{dV}{dt} \Big|_{r=2} = 4\pi (2)^2$$

$$= 4\pi \cdot 4$$

$$= 16\pi$$

$$\frac{dV}{dt} \Big|_{r=2} \approx 50.3 \text{ cm}^3/\text{year}$$

Sentence Answer:

When the radius of the gallstone is 2 cm, the volume is increasing by 50.3 cm³ per year.

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Gumball

#3) George is entering a chewed gumball contest. The radius of his spherical gumball is growing by $\frac{1}{2}$ centimeter per week. Find how rapidly the volume is increasing at the moment when the radius is 4 centimeters.

$$\frac{dr}{dt} = \frac{\frac{1}{2} \text{ cm}}{1 \text{ week}}$$

FIND $\frac{dV}{dt} \Big|_{r=4}$

$$\begin{aligned} r &= \text{radius (cm)} \\ t &= \text{time (week)} \\ V &= \text{volume (cm}^3\text{)} \end{aligned}$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dr} = 4\pi r^2 \left(\frac{dr}{dt}\right)$$

$$\frac{dV}{dr} = 4\pi r^2 \left(\frac{1}{2}\right)$$

$$\frac{dV}{dr} = 2\pi r^2$$

$$\frac{dV}{dr} \Big|_{r=4} = 2\pi(4)^2$$

$$= 2\pi(16)$$

$$= 32\pi$$

$$\frac{dV}{dr} \Big|_{r=4} \approx 100.5 \text{ cm}^3/\text{week}$$

Sentence Answer:

When the radius of the gum is 4 inches, the volume is increasing by 100.5 in³ per week

Pig Feet

#4) George's profit from selling x boxes pig feet is $P = 1000x - \frac{1}{2}x^2$ dollars. If sales are growing at the rate of 20 per day, find how rapidly profit is growing (in dollars per day) when 600 boxes have been sold.

$$\begin{aligned} x &= \text{sales (boxes of pig feet)} \\ P &= \text{profit } \$ \\ t &= \text{time (day)} \end{aligned}$$

$$\frac{dx}{dt} = \frac{20 \text{ boxes}}{1 \text{ day}}$$

FIND $\frac{dP}{dt} \Big|_{x=600}$

$$P = 1000x - \frac{1}{2}x^2$$

$$\frac{d}{dt}P = \frac{d}{dt}(1000x) - \frac{d}{dt}\left(\frac{1}{2}x^2\right)$$

$$\frac{dP}{dt} = 1000 \frac{dx}{dt} - x \frac{dx}{dt}$$

$$\frac{dP}{dt} = 1000(20) - x(20)$$

$$\frac{dP}{dt} = 20,000 - 20x$$

$$\frac{dP}{dt} \Big|_{x=600} = 20,000 - 20(600)$$

$$= 20,000 - 12,000$$

$$\frac{dP}{dt} \Big|_{x=600} = 8,000$$

Sentence Answer:

When 600 boxes of pigs feet have been sold, the profit is increasing by \$8000 per day.

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Tons of Feathers

#5) George's revenue from selling x tons of feathers is given as $R = 1000x - x^2$ dollars. If sales are increasing at the rate of 80 per day, find how rapidly revenue is growing (in dollars per day) when 400 tons have been sold.

x = sales (tons of feathers)
 R = Revenue \$
 t = time (days)

$$\frac{dx}{dt} = \frac{80 \text{ tons}}{1 \text{ day}}$$

FIND $\left. \frac{dR}{dt} \right|_{x=400}$

$$R = 1000x - x^2$$

$$\frac{d}{dt} R = \frac{d}{dt} (1000x) - \frac{d}{dt} (x^2)$$

$$\frac{dR}{dt} = 1000 \frac{dx}{dt} - 2x \frac{dx}{dt}$$

$$\frac{dR}{dt} = 1000(80) - 2x(80)$$

$$\frac{dR}{dt} = 80,000 - 160x$$

$$\left. \frac{dR}{dt} \right|_{x=400} = 80,000 - 160(400)$$

$$= 80,000 - 64,000$$

$$\left. \frac{dR}{dt} \right|_{x=400} = \$16,000/\text{day}$$

Sentence Answer:

When 400 tons of feathers have been sold, the revenue is increasing by \$16,000 per day.

Accidents

#6) The number of traffic accidents George's stench causes per year in a population p is predicted to be $T = 0.002p^{3/2}$. If the population is growing by 500 people per year, find the rate at which traffic accidents will be rising when the population is 40,000.

p = population (people)
 T = traffic accidents
 t = time (year)

$$\frac{dp}{dt} = \frac{500 \text{ people}}{1 \text{ year}}$$

FIND $\left. \frac{dT}{dt} \right|_{p=40,000}$

$$T = 0.002p^{3/2}$$

$$\frac{dT}{dt} = \frac{d}{dt} (0.002p^{3/2})$$

$$\frac{dT}{dt} = 0.003p^{1/2} \frac{dp}{dt}$$

$$\frac{dT}{dt} = 0.003\sqrt{p} (500)$$

$$\frac{dT}{dt} = 1.5\sqrt{p}$$

$$\left. \frac{dT}{dt} \right|_{p=40,000} = 1.5\sqrt{40,000}$$

$$= 1.5(200)$$

$$\left. \frac{dT}{dt} \right|_{p=40,000} = 300 \text{ Traffic accidents/year}$$

Sentence Answer:

When the pop'n is 40,000 people, the number of accidents George's Stench is causing is increasing by 300 accidents per year

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Carnival Slaying

#7) George is a carny and witnesses many types of crimes. The number of slayings at George's carnival of population p is expected to be $W = 0.003p^{4/3}$. If the population is growing by 1000 people per year, find the rate at which the number of carnival slayings will be increasing when the population is 1,000,000.

P = population of carnival
 W = # of slayings
 t = time in years

$$\frac{dP}{dt} = \frac{1000 \text{ people}}{1 \text{ year}}$$

$$\frac{dW}{dt} \Big|_{P=1,000,000}$$

$$W = 0.003p^{4/3}$$

$$\frac{d}{dt}(W) = \frac{d}{dt}(0.003p^{4/3})$$

$$\frac{dW}{dt} = 0.004p^{1/3} \frac{dP}{dt}$$

$$\frac{dW}{dt} = 0.004\sqrt[3]{P} (1000)$$

$$\frac{dW}{dt} = 4\sqrt[3]{P}$$

$$\frac{dW}{dt} \Big|_{P=1,000,000} = 4\sqrt[3]{1,000,000}$$

$$= 4(1000)$$

$$\frac{dW}{dt} \Big|_{P=1,000,000} = 4000 \text{ slayings per year}$$

Sentence Answer:

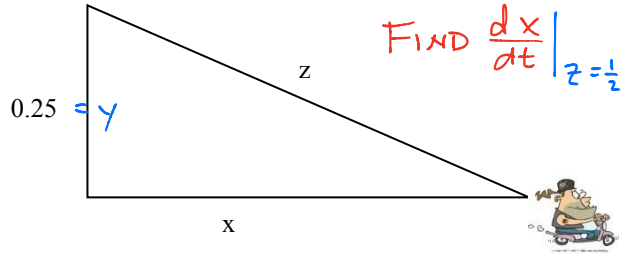
When the carnival pop'n is 1,000,000, the number of carnival slayings is increasing by 4000 slayings per year.

Speeding

#8) A traffic patrol helicopter is stationary a quarter of a mile directly above a highway, as shown in the diagram below. Its radar detects George's moped whose line-of-sight distance from the helicopter is half a mile and is increasing at the rate of 57 mph. Is the moped exceeding the highway's speed limit of 60 mph?



$$\frac{dz}{dt} = \frac{57 \text{ mi}}{1 \text{ hr}}$$



when $z = 0.5$

$(.25)^2 + x^2 = (.5)^2$
 $.0625 + x^2 = .25$
 $x^2 = .1875$
 $x \approx .433$

$$x^2 + y^2 = z^2$$

$$x^2 + (.25)^2 = z^2$$

$$x^2 + 0.0625 = z^2$$

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(0.0625) = \frac{d}{dt}(z^2)$$

$$2x \frac{dx}{dt} = 2z \frac{dz}{dt}$$

$$\frac{dx}{dt} = \frac{z}{x} \frac{dz}{dt}$$

$$\frac{dx}{dt} = \frac{z}{x} (57)$$

$$\frac{dx}{dt} \Big|_{\substack{z=1/2 \\ x=.433}} \approx \frac{(1/2)}{(.433)} 57$$

$$\approx 65.8 \text{ mph}$$

Sentence Answer:

When George is $\frac{1}{2}$ mile from the chopper, he is breaking the speed limit by traveling at 65.8 miles per hour.

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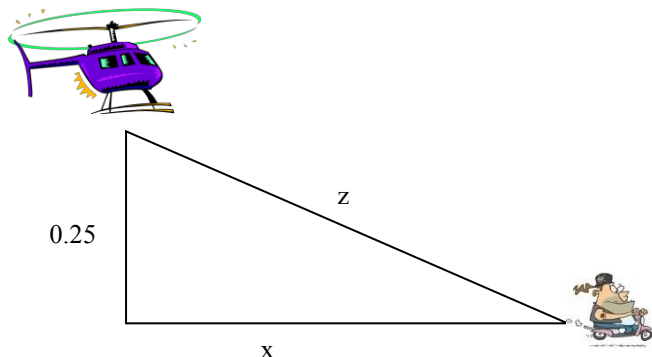
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- #1) At the moment the radius is 3 inches, the volume is decreasing by $72\pi \approx 226 \text{ in}^3$ in per hour.
- #2) When the radius is 2 mm, the volume is growing at $16\pi \approx 50.27 \text{ mm}^3$ per minute.
- #3) When the radius is 4 cm, the volume of the tumor is growing at $32\pi \approx 101 \text{ cm}^3$ per week.
- #4) When 600 units have been sold, the profit is growing by \$8000 per day.
- #5) When 400 units have been sold, the revenue is growing by \$16,000 per day.
- #6) When the population is 40,000 people, traffic accidents will be rising by 300 accidents per year.
- #7) When the population is 1,000,000 people, the number of carnival slayings is increasing by 400 cases per year.
- #8) The car is traveling at 65.8 mph, so yes the car is speeding.

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#8)



Facts:

z = line of sight distance in miles

x = horizontal distance in miles

$$\frac{dz}{dt} = 57 \text{ miles/hour}$$

Find $\frac{dx}{dt}$ evaluated at $z = .5$ miles

From Pythagoras

$$\begin{aligned} x^2 + 0.25^2 &= z^2 \\ x^2 + 0.25^2 &= 0.5^2 \\ x^2 + 0.0625 &= 0.25 \\ x^2 &= .1875 \\ x &\approx 0.433 \end{aligned}$$

From Pythagoras

$$x^2 + 0.25^2 = z^2$$

$$\frac{d}{dt} x^2 + \frac{d}{dt} 0.25^2 = \frac{d}{dt} z^2$$

Found $\frac{d}{dt}$

$$2x \frac{dz}{dt} + 0 = 2z \frac{dz}{dt}$$

Simplified

$$2x \frac{dz}{dt} = 2z \frac{dz}{dt}$$

Simplified

$$\frac{dz}{dt} = \frac{2z}{2x} \frac{dz}{dt}$$

Solving for $\frac{dz}{dt}$, I divided by $2x$

$$= \frac{z}{x} \frac{dz}{dt}$$

The 2's canceled, because $2 / 2 = 1$

$$\frac{dz}{dt} \text{ evaluated } \approx \frac{0.5}{.433} 57$$

Substituted $\frac{dz}{dt} = 57, z = .5, x \approx 0.433$

$$\approx \frac{0.5(57)}{.433}$$

Simplified

$$\approx 65.8 \text{ miles per hour}$$

Simplified

Yes, the moped is exceeding the highway's speed of 60 mph. George is traveling at a rate of approximately 65.8 mph.