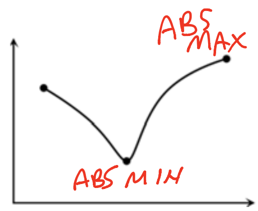
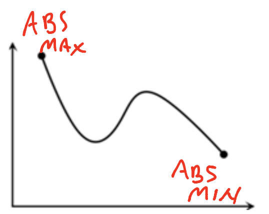
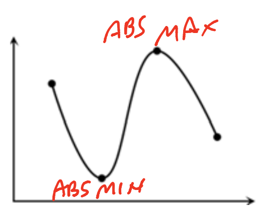
Graphing & Basic Optimization
5.2 – Optimization

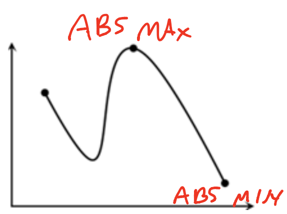
Optimization: to find the maximum or minimum value a function
Absolute maximum value: the largest value of the function on its domain. (highest point on the graph)
Absolute minimum value: the smallest value of the function on its domain. (lowest point on the graph)

Closed Interval
In a closed interval of a continuous function, the **absolute** maximum and minimum are guaranteed to exist.







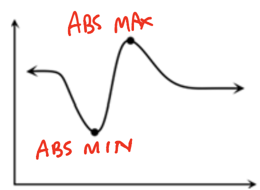


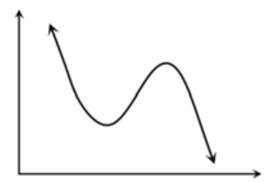
<u>To optimize a continuous function *f* on [a, b]:</u>

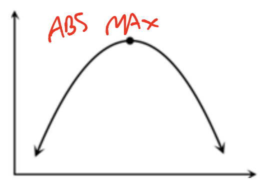
1. Find all critical values of *f* in [a, b]
2. Identify the end values
3. Evaluate $f(CV)$ and $f(EV)$

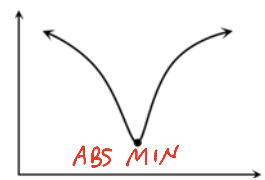
The largest and smallest y-values found in step 3 will be the absolute maximum and minimum values of *f* on [a, b].

Open Interval
In an open interval of a function, both absolute extreme values may exist, or one or both may fail to exist.









Second-Derivative Test

Determining whether a twice-differentiable function has a relative maximum or minimum at a critical value can be determined by concavity.

Second-Derivative Test for *Relative* Extremes

If $x = c$ is a critical value of *f* at which f'' is defined, then

$f''(c) > 0$ means that *f* has a relative minimum at $x = c$.

$f''(c) < 0$ means that *f* has a relative maximum at $x = c$.

$f''(c) = 0$ is inconclusive. Second-Derivative Test failed.

To use the second-derivative test, we first find all the critical values, substituting each into the second derivative and determining the sign result. A *positive* result means a *minimum* at the critical value, and a *negative* result means a *maximum*. (If the second derivative is zero, then the test is inconclusive, and you should use the first-derivative test or make a sign diagram for *f* ‘.)

If there is only one critical point, then the Second-Derivative Test can be used to find the Absolute Extreme Point.

5.2 – Optimization

Ex A: Optimizing on a Closed Interval

Find the absolute extreme values of $f(x) = x^3 - 9x^2 + 15x$ on $[0, 4]$ and then graph.

CV

1. $f'(x) = 3x^2 - 18x + 15$

$0 = 3(x^2 - 6x + 5)$

$0 = 3(x-5)(x-1)$

ZOH

$0 \neq 3$ | $0 = x - 5$, $5 = x$ | $0 = x - 1$, $1 = x$

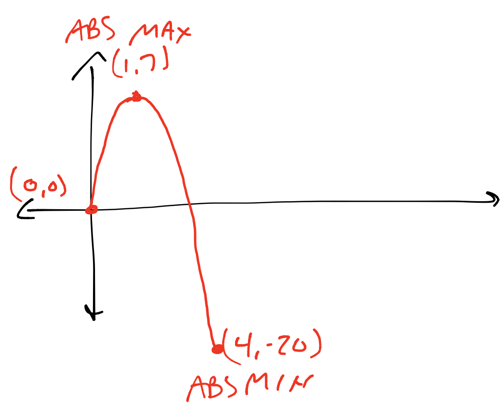
CV: $x = 1, \cancel{5}$ Not $\in [0,4]$

2. EV: $x = 0, 4$

3. $f(1) = 7$
$f(0) = 0$
$f(4) = -20$

CP: $(1,7)$
EP: $(0,0)$, $(4,-20)$

4. MAX $(1,7)$
MIN $(4,-20)$



ABS MIN: $(4,-20)$

ABS MAX $(1,7)$

Pro Tips

If the function is continuous and the interval is closed, then both extreme values exist.

The endpoints, known as EP, are the beginning and end of the domain interval.

#1) Find CVs by solving $f' = 0$. Throw out any critical values not in the domain.

#2) Identify the EV (end values).

#3) Find the CP and EP by evaluating $f(CV)$ and $f(EV)$.

#4) Identify the Absolute Minimum and Absolute Maximum on the interval.

Ex B: Optimizing on an Open Interval

The value of a timber forest after t years is $V(t) = 48\sqrt{t} - 6t$ thousand dollars (for $t > 0$). Find when its value is maximized.

V = $ thousand
t = years

CV

1. $V'(t) = 24t^{-\frac{1}{2}} - 6$

$0 = \frac{24}{\sqrt{t}} - 6$

$6 = \frac{24}{\sqrt{t}}$

$6\sqrt{t} = 24$

$\sqrt{t} = 4$

$t = 16$

CV: $t = 16$

2. $V''(t) = -12t^{-3/2}$

$V''(t) = \frac{-12}{\sqrt{t^3}}$

$V''(16) = \frac{-}{+} = -$, concave Down, MAX

CP

3. $V(16) = \$96$ thousand

In 16 years, the value of the timber will be maximized with a value of $96,000.

Pro Tips

#1) Find CV by solving $f' = 0$.

#2) Since there is only one CP, we can use the second-derivative test. To do so, find $f''(CV)$.

If $f''(CV) > 0$, the we have an absolute min

If $f''(CV) < 0$, then we have an absolute max.

#3) Find the CP by evaluating $f(CV)$.

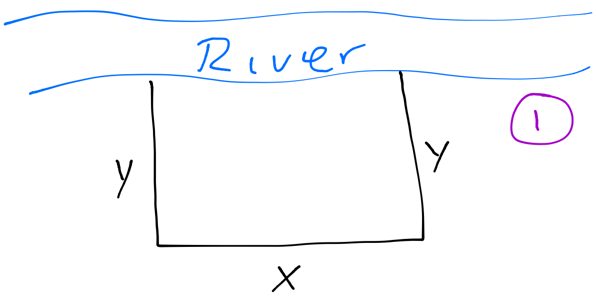
(Steps #2 and #3 are interchangeable)

Graphing & Basic Optimization

5.2 – Optimization

Ex C: Maximizing the Area of an Enclosure

A man lives in a white van down by the river. While spelunking he finds 1000 feet of pristine fence. He decides to build a rectangular enclose along the river to mark his territory. If the side along the river needs no fence, find the dimensions that make his territory as large as possible. Also find the maximum area.



1

$x = \parallel$ to river in feet

$y = \perp$ to river in feet

A : Area

P = Perimeter

2 $A = x \cdot y$

3 $A = (1000 - 2y)\,y$

$A(y) = 1000y - 2y^2$

2 $P = x + y + y$

$1000 = x + 2y$

$1000 - 2y = x$

4 $A'(y) = 1000 - 4y$

$0 = 1000 - 4y$

$4y = 1000$

$y = 250$

$A''(y) = -4$

$A''(y) = (-)$, concave down, MAX

5 $1000 - 2(250) = x$

$1000 - 500 = x$

$500 = x$

6 $A = x\,y$

$= (500)(250)$

$A = 125{,}000 \text{ ft}^2$

In order to maximize the area at 125,000 ft^2 the farmer needs to make the sides parallel to the river 500 ft and the side perpendicular to the river 250 ft.

Pro Tips

#1) Draw a picture and define any variables you may use.

#2) Based on picture, write equations for area and perimeter.

#3) Use the perimeter and area equations to write the Area function in one variable.

#4) Find the maximum of the Area function. (Solve $f' = 0$. Then determine if max or min by second-derivative test.

#5) Substitute the answer from #4 into the perimeter equation to find the other dimension.

#6) Use the dimensions in the area equation to find the maximum area.

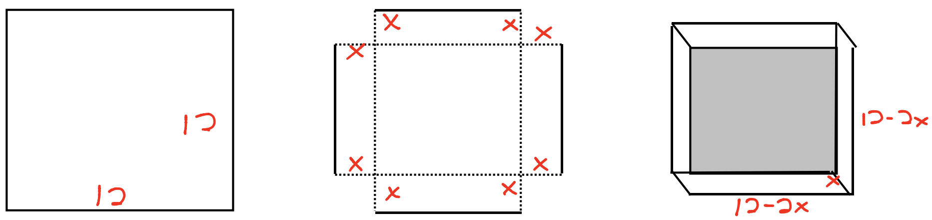
Ex D: Maximizing the Volume of a Box

An open-top box is to be made from a square sheet of metal 12 inches on each side by cutting a square from each corner and folding up the sides, as in the diagram below. Find the volume of the largest box that can be made.

x = length of cut in inches

V = volume

#1



Square Sheet Corners Removed Sides folded up

② $V = \text{Area}_{base} \cdot \text{height}$

$V = (12-2x)(12-2x)x$

$V = (144 - 48x + 4x^2)x$

$V = 4x^3 - 48x^2 + 144x$

Domain = $(0, 6)$

③ $V'(x) = 12x^2 - 96x + 144$

$0 = 12(x^2 - 8x + 12)$

$0 = 12(x-6)(x-2)$

$0 \neq 12$, $0 = x-6$, $6 = x$; $0 = x-2$, $2 = x$

CV: $x = 2, \cancel{6}$ (Not in Domain)

$V''(x) = 24x - 96$

$V''(x) = 24(x-4)$

$V''(2) = (+)(-) = -$, Concave DN

MAX

④ CP

$\boxed{V(2) = 128 \text{ in}^3}$

In order to maximize the volume of the box to 128 in^3, there needs to be 2" cut off the corners giving dimensions of 2" by 8" by 8"

Pro Tips

#1) Draw a picture and define any variables you may use.

#2) Based on picture, write an equation for volume.

#3) Find the max of the Volume function. (Solve $f' = 0$. Then determine if max or min by second-derivative test.

#4) Find Volume by evaluating F(CV).