

Graphing & Basic Optimization

Review Chapter 5

Graph the following function by hand by making a first derivative sign diagram and then a second derivative sign diagram.

#1) $f(x) = x^2(x^2 - 4)$

$$f(x) = x^4 - 4x^2$$

CV

$$f'(x) = 4x^3 - 8x$$

$$0 = 4x(x^2 - 2)$$

ZON

$$\begin{cases} 0 = 4x \\ 0 = x^2 - 2 \end{cases}$$

$$\begin{cases} 0 = x \\ 2 = x^2 \end{cases}$$

$$\pm\sqrt{2} = x$$

CV: $x = -\sqrt{2}, 0, \sqrt{2}$

CV

$$f''(x) = 12x^2 - 8$$

$$0 = 12x^2 - 8$$

$$0 = 4(3x^2 - 2)$$

ZON

$$\begin{cases} 0 = 3x^2 - 2 \\ 2 = 3x^2 \end{cases}$$

$$\frac{2}{3} = x^2$$

$$\pm\sqrt{\frac{2}{3}} = x$$

CP

$$f(-\sqrt{2}) = -4$$

$$f(0) = 0$$

$$f(\sqrt{2}) = -4$$

CP: $(-\sqrt{2}, -4), (0, 0), (\sqrt{2}, -4)$

CP

$$f(-\sqrt{\frac{2}{3}}) = -2.2$$

$$f(\sqrt{\frac{2}{3}}) = -2.2$$

CP: $(-\sqrt{\frac{2}{3}}, -2.2), (\sqrt{\frac{2}{3}}, -2.2)$

$$f'(x) = 4x(x^2 - 2)$$

$$\begin{array}{c|ccc} (+)(-) & - & (+)(-) & + \\ f' < 0 & | & f' > 0 & | \\ \hline -3 & f'(-\sqrt{2}) = 0 & f'(0) = 0 & f'(\sqrt{2}) = 0 & 3 \end{array}$$

$$\begin{array}{ccccc} & MAX & & MIN & \\ (-\sqrt{2}, -4) & \nearrow & (0, 0) & \searrow & (\sqrt{2}, -4) \\ MIN & & & & MIN \end{array}$$

$$f''(x) = 4(3x^2 - 2)$$

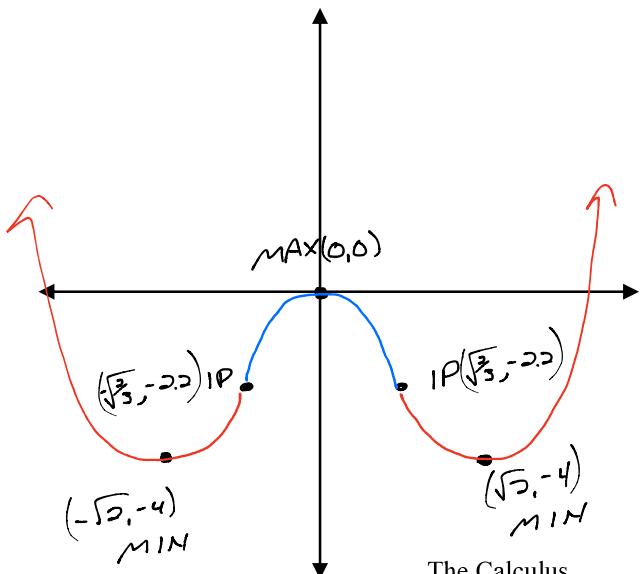
$$\begin{array}{c|ccc} (+)(+) & + & (+)(-) & - \\ f'' > 0 & | & f'' < 0 & | \\ \hline -100 & f''(-\sqrt{\frac{2}{3}}) = 0 & 0 & f''(\sqrt{\frac{2}{3}}) = 0 & 100 \end{array}$$

CCU **IP** **CCD** **IP** **CCU**

$$(-\sqrt{\frac{2}{3}}, -2.2) \quad (\sqrt{\frac{2}{3}}, -2.2)$$

Y-int

$$f(0) = 0$$



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Review Chapter 5

Graph the following function by hand by making a first derivative sign diagram and then a second derivative sign diagram.

#2) $f(x) = \frac{10x^2}{x^2 - 9}$

RATIONAL

$$f(x) = \frac{10x^2}{(x-3)(x+3)}$$

NO HOLES

V A
$(x-3)(x+3) = 0$
$x-3 = 0 \quad x+3 = 0$
$x = 3 \quad x = -3$

HA
$n? d$
$d = 2, \text{ so } y = 10$

NO SA

CV

$$f'(x) = \frac{(10x^2)'(x^2 - 9) - 10x^2(x^2 - 9)'}{(x^2 - 9)^2}$$

$$\therefore = \frac{20x(x^2 - 9) - 10x^2(2x)}{(x^2 - 9)^2}$$

$$\therefore = \frac{20x^3 - 180x - 20x^3}{(x^2 - 9)^2}$$

$$\therefore = \frac{-180x}{(x^2 - 9)^2}$$

ZON

$0 = -180x$
$0 = x$
$CV: x = 0$

<u>ZOD</u>
$0 = (x^2 - 9)^2$
$0 = x^2 - 9$
$9 = x^2$
$\pm 3 = x$
$CV: x = -3, 3$

y-int

$$f(0) = 0$$

CP

$f(-3) = \text{und}$
vA
$f(0) = \text{und}$
$f(3) = \text{und}$
vA

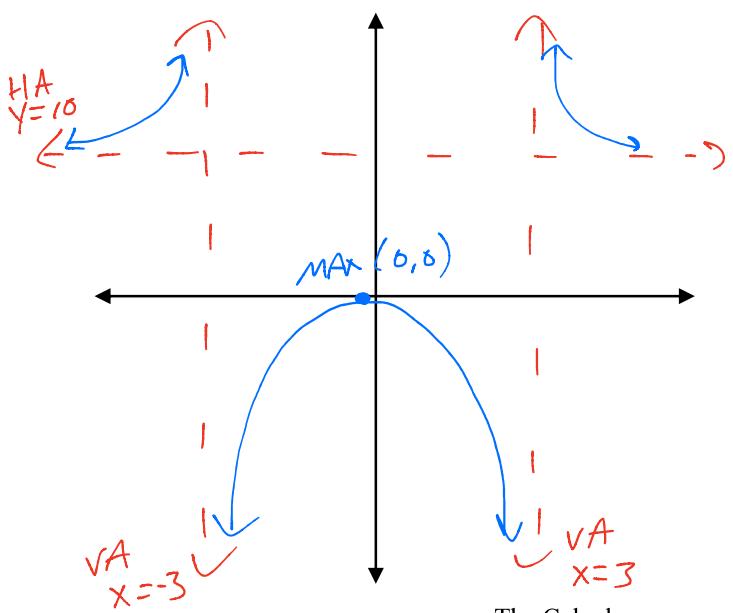
$$CP: (0, 0)$$

$$f'(x) = \frac{-180x}{(x^2 - 9)^2}$$

$$\begin{array}{c|c|c|c|c} (-\infty, -3) & (-3, 0) & (0, 3) & (3, \infty) \\ \hline f' > 0 & f' > 0 & f' < 0 & f' < 0 \end{array}$$

$$\begin{array}{c|c|c|c} (-\infty, -3) & (-3, 0) & (0, 3) & (3, \infty) \\ \hline f'(\text{und}) & f'(0) = 0 & f'(3) = \text{und} & \end{array}$$

$\text{MAX}(0, 0)$



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Review Chapter 5

Graph the following function by hand by making a first derivative sign diagram and then a second derivative sign diagram.

#3) $f(x) = x^3 + 3x^2 + 3x + 6$

$f'(x) = 3x^2 + 6x + 3$

$$0 = 3(x^2 + 2x + 1)$$

$$0 = 3(x+1)^2$$

ZON

$$(x+1)^2 = 0$$

$$x+1 = 0$$

$$x = -1$$

CV: $x = -1$

$f(-1) = 5$

$(-1, 5)$

$$f'(x) = 3(x+1)^2$$

$$\begin{array}{c|c} f' > 0 & ; \\ \hline f'(-1) = 0 & \end{array}$$

$f' > 0$

Neither
 $(-1, 5)$

y_{int}

$f(0) = 6$

$f''(x) = 6x + 6$

$$0 = 6(x+1)$$

ZON

$$x+1 = 0$$

$$x = -1$$

CV: $x = -1$

$f(-1) = 5$

$(-1, 5)$

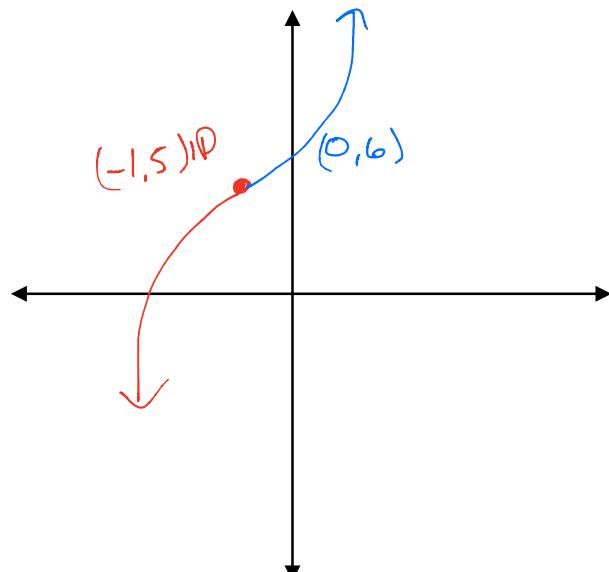
$$f''(x) = 6(x+1)$$

$$\begin{array}{c|c} f'' < 0 & ; \\ \hline f''(-1) = 0 & \end{array}$$

$(+) (-) = -$ $(+) (+) = +$

$f'' > 0$

CCD CP CCO
 $(-1, 5)$



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Graph the following function by hand by making a first derivative sign diagram and then a second derivative sign diagram.

#4) $f(x) = x^3(x - 4)$

$$f'(x) = x^4 - 4x^3$$

CV

$$f'(x) = 4x^3 - 12x^2$$

$$0 = 4x^2(x - 3)$$

ZON

$$\begin{cases} 0 = 4x^2 \\ 0 = x^2 \end{cases} \quad \begin{cases} 0 = x - 3 \\ 3 = x \end{cases}$$

$$0 = x$$

$$\text{CV: } x = 0, 3$$

CV

$$f''(x) = 12x^2 - 24x$$

$$0 = 12x(x - 2)$$

ZON

$$\begin{cases} 12x = 0 \\ 0 = x \end{cases} \quad \begin{cases} 0 = x - 2 \\ 2 = x \end{cases}$$

$$0 = x$$

$$\text{CV: } x = 0, 2$$

CP

$$f(0) = 0$$

$$f(3) = -27$$

$$\text{CP: } (0, 0), (3, -27)$$

CP

$$f(0) = 0$$

$$f(2) = -16$$

$$\text{CP: } (0, 0), (2, -16)$$

$$f'(x) = 4x^2(x - 3)$$

$(+)(-) = -$	$(+)(-) = -$	$(+)(+) = +$
$f' < 0$	$f' < 0$	$f' > 0$
$\leftarrow -1$	$f'(0) = 0$	$1 \rightarrow 4$

$\xrightarrow{(0, 0)} \text{neutral}$ $\xrightarrow{(3, -27)} \text{MIN}$

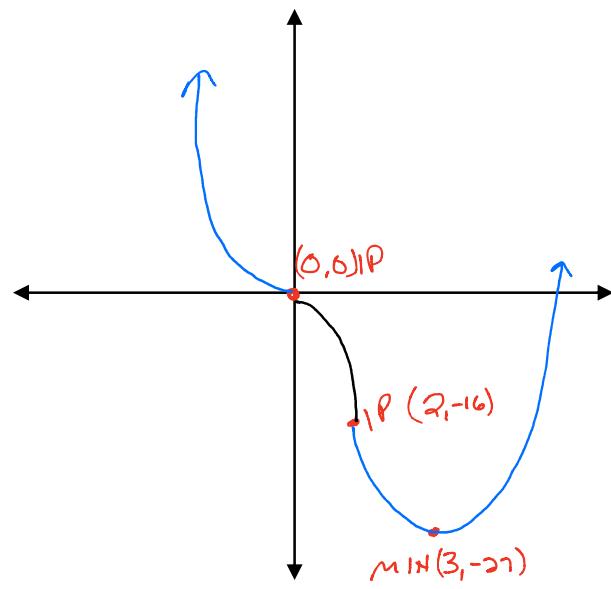
$$f''(x) = 12x(x - 2)$$

$(-)(-) = +$	$(+)(-) = -$	$(+)(+) = +$
$f'' > 0$	$f'' < 0$	$f'' > 0$
$\leftarrow -1$	$f''(0) = \text{und}$	$1 \rightarrow 3$

$\xleftarrow{\text{CCW}} \text{IP } (0, 0)$ $\xleftarrow{\text{CCD}} \text{IP } (2, -16)$ $\xrightarrow{\text{CCW}}$

$y\text{-int}$

$$f(0) = 0$$



Graphing & Basic Optimization

Review Chapter 5

Find the absolute extreme points of each function on the given interval.

$$\#5) f(x) = x^3 - 6x^2 + 22 \text{ on } [-2, 2]$$

$$f'(x) = 3x^2 - 12x$$

$$0 = 3x^2 - 12x$$

$$0 = 3x(x - 4)$$

$$\begin{aligned} 0 &= 3x \\ 0 &= x-4 \end{aligned}$$

$$(Cv: x=0, 4 \notin [-2, 2])$$

$$(Ev: x=-2, 2)$$

$$f(0) = 22 \text{ MAX}$$

$$f(-2) = -10 \text{ MIN}$$

$$f(2) = 6$$

$$(CP: (0, 22))$$

$$(EP: (-2, -10), (2, 6))$$

$$\#6) f(x) = x^4 - 4x^3 + 4x^2 \text{ on } [0, 3]$$

$$f'(x) = 4x^3 - 12x^2 + 8x$$

$$0 = 4x(x^2 - 3x + 2)$$

$$0 = 4x(x-2)(x-1)$$

$$0 = 4x \quad \left\{ \begin{array}{l} 0 = x-2 \\ 0 = x-1 \end{array} \right. \quad \left\{ \begin{array}{l} 0 = x \\ 2 = x \\ 1 = x \end{array} \right.$$

$$(Cv: x=0, 1, 2)$$

$$(Ev: x=0, 3)$$

$$f(0) = 0 \text{ MIN}$$

$$f(1) = 1$$

$$f(2) = 0 \text{ MIN}$$

$$f(3) = 9 \text{ MAX}$$

$$(CP: (0, 0), (1, 1), (2, 0))$$

$$(EP: (0, 0), (3, 9))$$

$$\text{Absolute Max: } (0, 22)$$

$$\text{Absolute Min: } (-2, -10)$$

$$\text{Absolute Max: } (3, 9)$$

$$\text{Absolute Min: } (0, 0), (2, 0)$$

Graphing & Basic Optimization

Review Chapter 5

#7) The fuel economy of an average compact car in Gnaden is $E(x) = -0.015x^2 + 1.14x + 8.3$, where x is the driving speed (in miles per hour $20 \leq x \leq 60$). At what speed is fuel economy the greatest? What is the greatest mpg?

$$E'(x) = -0.030x + 1.14$$

$$0 = -0.030x + 1.14$$

$$0.030x = 1.14$$

$$\text{cv: } x = 38$$

$$\text{Ev: } x = 20, 40, 60$$

$$E(38) = 29.86 \text{ MAX}$$

$$E(20) = 25.1$$

$$E(60) = 22.7$$

#8) Two Wheel Deals finds that it costs \$70 to manufacture each bicycle, and fixed costs are \$100 per day. The price function is $p(x) = 270 - 10x$, where p is the price (in dollars) at which exactly x bicycles will be sold. Find the quantity Two Wheel Deals should produce and the price it should charge to maximize profit. Also find the maximum profit.

$$C(x) = 70x + 100$$

$$R(x) = p(x) \text{ qty} \\ = (270 - 10x)x$$

$$R(x) = 270x - 10x^2$$

$$P(x) = R(x) - C(x) \\ = (270x - 10x^2) - (70x + 100)$$

$$P(x) = -10x^2 + 200x - 100$$

$$P'(x) = -20x + 200$$

$$0 = -20x + 200$$

$$20x = 200$$

$$x = 10$$

$$P''(x) = -20$$

$$P''(10) = \text{neg, CCD, MAX}$$

$$p(10) = 270 - 10(10)$$

$$= 270 - 100$$

$$p(10) = 170$$

$$P(10) = -10(10)^2 + 200(10) - 100$$

$$= -10(100) + 2000 - 100$$

$$= -1000 + 2000 - 100$$

$$= 1000 - 100$$

$$P(10) = 900$$

Quantity that should be produced = 10

Price to be sold = \$170

Maximum Profit = \$900

Sentence Answer

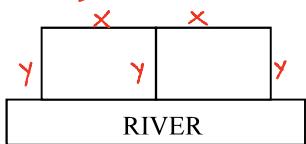
At 38 mph the car is most efficient at 29.86 mpg.

Sentence Answer

Two Wheel Deals should set their price at \$170 per bike and sell 10 bikes. This will max their profit at \$900.

Graphing & Basic Optimization Review Chapter 5

#9) A farmer wants to make two identical rectangular enclosures along a straight river, as in the diagram shown below. If he has 600 yards of fence, and if the sides along the river need no fence, what should be the dimensions of each enclosure if the total area is to be maximized?



$x = \text{length // to river of each enclosure}$
 $y = \text{length } \perp \text{ to river}$
 $A = \text{Area}$
 $P = \text{Perimeter}$

$$A_{\text{Total}} = 2xy$$

$$A = 2x(200 - \frac{2}{3}x)$$

$$A = 400x - \frac{4}{3}x^2$$

$$A' = 400 - \frac{8}{3}x$$

$$0 = 400 - \frac{8}{3}x$$

$$\frac{8}{3}x = 400$$

$$x = \frac{400}{\frac{8}{3}} \cdot 3$$

$$x = 150$$

$$A''(x) = -\frac{8}{3}$$

$$A''(150) = \text{neg, CD, max}$$

$$P_{\text{Fence}} = 2x + 3y$$

$$600 = 2x + 3y$$

$$600 - 2x = 3y$$

$$200 - \frac{2}{3}x = y$$

$$200 - \frac{2}{3}(150) = y$$

$$200 - 100 = y$$

$$100 = y$$

$$A_{(150, 100)} = 2(150)(100)$$

$$A_{(150, 100)} = 30,000 \text{ ft}^2$$

Side parallel to the river of one enclosure = 150 yards

Side perpendicular to the river of one enclosure = 100 yards

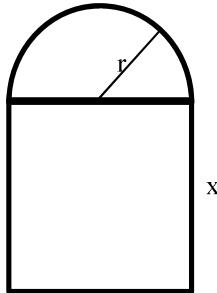
Total Maximized Area = 30,000 yards²

Sentence Answer

To max the total area to 30,000 yd², the side parallel to the river (of each enclosure) should be 150 yards and the sides \perp to river should be 100 yds.

Graphing & Basic Optimization Review Chapter 5

#10) If the perimeter of the window below is to be 18 feet, find the dimensions (x and r) that maximize the area of the window. (hint: The perimeter is 3 sides of a rectangle and half a circle.)



$$A = \text{Area}$$

$$P = \text{perimeter}$$

$$A_{\text{total}} = \frac{1}{2}A_O + A_{\text{Rec}}$$

$$= \frac{1}{2}\pi r^2 + 2r(x)$$

$$A = \frac{1}{2}\pi r^2 + 2rx$$

$$A = \frac{1}{2}\pi r^2 + 2r(9 - r - \frac{\pi}{2}r)$$

$$A = \frac{1}{2}\pi r^2 + 18r - 2r^2 - \pi r^2$$

$$A' = \pi r + 18 - 4r - 2\pi r$$

$$0 = 18 + \pi r - 4r - 2\pi r$$

$$-18 = r(\pi - 4 - 2\pi)$$

$$\frac{-18}{\pi - 4 - 2\pi} = r$$

$$2.5 \approx r$$

$$P = 2x + 2r + \pi r$$

$$18 = 2x + 2r + \pi r$$

$$18 - 2r - \pi r = 2x$$

$$9 - r - \frac{\pi}{2}r = x$$

$$9 - (2.5) - \frac{\pi}{2}(2.5) = x$$

$$6.5 - 1.25\pi = x$$

$$2.6 \approx x$$

$$A = \frac{1}{2}\pi r^2 + 2rx$$

$$A(2.5, 2.6) = \frac{1}{2}\pi(2.5)^2 + 2(2.5)(2.6)$$

$$= 3.125\pi + 13$$

$$A(6.5, 3.6) \approx 22.8 \text{ ft}^2$$

$$A'' = \pi - 4 - 2\pi$$

$$A''(2.5) = \text{neg. ccd, MAX}$$

$$x = 2.5 \text{ feet}$$

$$r = 2.6 \text{ feet}$$

$$\text{Total Maximized Area} = 22.8 \text{ ft}^2$$

Sentence Answer: To max area of window to 22.8 ft^2 , r and x should be 2.5 ft and 2.6 ft , respectively.