3.2 Product and Quotient Rule

PRACTICE

Find the derivative of the following.

1. $f(x) = \frac{5x-2}{x^2+1}$

$$f' = \frac{(5x-2)'(x^2+1)-(5x-2)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{(5)(x^2+1)-(5x+2)(2x)}{(x^2+1)^2}$$

$$= \frac{5x^2+5-10x^2+4x}{(x^2+1)^2}$$

$$f' = \frac{-5x^2+4x+5}{(x^2+1)^2}$$

2. $g(x) = (2x+1)(x^3-1)$

$$g' = (2x+1)'(x^3-1)+(2x+1)(x^3-1)'$$

$$= (2)(x^3-1)+(2x+1)(3x^2)$$

$$= 2x^3-2+6x^3+3x^2$$

$$g' = 8x^3+3x^2-2$$

3. $y = (3x^2-2x)(x^2+3x-4)$

$$\frac{dy}{dx} = \frac{d}{dx}(3x^2-2x)\cdot(x^3+3x-4)+(3x^2-2x)\cdot\frac{d}{dx}(x^3+3x-4)$$

$$\frac{dy}{dx} = \underbrace{(6x-2)}_{\text{binomial}}\cdot\underbrace{(x^3+3x-4)}_{\text{Trinomial}}+(3x^2-2x)(3x^2+3)$$

Don't need to simplify

4. $h(x) = \frac{6x^2+3x-5}{3x}$

$$h(x) = \frac{6x^2}{3x}+\frac{3x}{3x}-\frac{5}{3x}$$

$$h(x) = 2x+1-\frac{5}{3}x^{-1}$$

$$h'(x) = 2+\frac{5}{3}x^{-2}$$

$$h'(x) = 2+\frac{5}{3x^2}$$

5. $f(t) = \frac{t+1}{\sqrt{t}}$

$$f(t) = \frac{t}{t^{\frac{1}{2}}}+\frac{1}{t^{\frac{1}{2}}}$$

$$f(t) = t^{\frac{1}{2}}+t^{-\frac{1}{2}}$$

$$f'(t) = \frac{1}{2}t^{-\frac{1}{2}}-\frac{1}{2}t^{-\frac{3}{2}}$$

$$f'(t) = \frac{1}{2\sqrt{t}}-\frac{1}{2\sqrt{t^3}}$$

6. $f(r) = r^2(5r^3+3)$

$$f(r) = 5r^5+3r^2$$

$$f'(r) = 25r^4+6r$$

Find the derivatives of the following.

7. $y = \frac{x}{x-1}$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(x)\cdot(x-1)-x\cdot\frac{d}{dx}(x-1)}{(x-1)^2} = \frac{(1)(x-1)-x(1)}{(x-1)^2} = \frac{x-1-x}{(x-1)^2}$$

$$= \frac{-1}{(x-1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(-1)'(x-1)^2-(-1)(x^2-2x+1)'}{[(x-1)^2]^2}$$

$$= \frac{0(x-1)^2+(2x-2)}{(x-1)^4}$$

$$\frac{d^2y}{dx^2} = \frac{2x-2}{(x-1)^4}$$

8. $y = x^{-2}(ex^3+3)$

$$= ex+3x^{-2}$$

$$y' = e-6x^{-3}$$

$$y'' = 18x^{-4}$$

Given $f(x) = (x^2 - 5)(3x + 2)$, find the following.

9. $f'(2) = 29$

$f'(x) = (x^2-5)'(3x+2) + (x^2-5)(3x+2)'$

$= (2x)(3x+2) + (x^2-5)(3)$

$= 6x^2 + 4x + 3x^2 - 15$

$f'(x) = 9x^2 + 4x - 15$

$f'(2) = 9(2)^2 + 4(2) - 15$

$= 9(4) + 8 - 15$

$= 36 - 7$

$f'(2) = 29$

10. Find the slope of $f(x)$ at $x = -3$.

$f'(x) = 9x^2 + 4x - 15$

$f'(-3) = 9(-3)^2 + 4(-3) - 15$

$= 9(9) - 12 - 15$

$= 81 - 27$

$f'(-3) = 54$

11. What is the slope of the tangent line of $f(x)$ at the point (4, 48) ?

$f'(x) = 9x^2 + 4x - 15$

$f'(4) = 9(4)^2 + 4(4) - 15$

$= 9(16) + 16 - 15$

$= 144 + 1$

$f'(4) = 145$

Is the slope of the tangent line positive, negative, or zero at the given point?

12. $f(x) = \dfrac{2 - \frac{1}{x}}{x-3}$ at $x = 4$

$f'(x) = \dfrac{(2 - x^{-1})'(x-3) + (2 - \frac{1}{x})(x-3)'}{(x-3)^2}$

$= \dfrac{(x^{-2})(x-3) + (2 - \frac{1}{x})(1)}{(x-3)^2}$

$= \dfrac{\frac{1}{x} - \frac{3}{x^2} + 2 - \frac{1}{x}}{(x-3)^2}$

$f'(x) = \dfrac{-\frac{3}{x^2} + 2}{(x-3)^2}$

Sign analysis

$f'(4) = \dfrac{\frac{-3}{(4)^2} + 2}{+}$

$f'(4) = \dfrac{+}{+} = +$

The slope is positive

13. $g(x) = (x + 1)^2$ at $x = -4$

$g(x) = x^2 + 2x + 2$

$g'(x) = 2x + 2$

$g'(x) = 2(x+1)$

Sign analysis

$g'(-4) = (+)(-)$

$g'(-4) = -$

The slope is negative

Determine the x-values (if any) at which the function has a horizontal tangent line.

14. $f(x) = \dfrac{4x^3 - 10x^2}{2x}$

$f(x) = 2x^2 - 5x$

$f'(x) = 4x - 5$

$0 = 4x - 5$

$5 = 4x$

$\frac{5}{4} = x$

There is a horizontal tangent line at $x = \frac{5}{4}$

15. $g(x) = \dfrac{x^2}{x+1}$

$g'(x) = \dfrac{(x^2)'(x+1) - x^2(x+1)'}{(x+1)^2}$

$= \dfrac{2x(x+1) - x^2(1)}{(x+1)^2}$

$= \dfrac{2x^2 + 2x - x^2}{(x+1)^2}$

$g'(x) = \dfrac{x^2 + 2x}{(x+1)^2}$

$0 = \dfrac{x^2 + 2x}{(x+1)^2}$

ZON: $0 = x^2 + 2x$, $0 = x(x+2)$, $0 = x$ or $x + 2 = 0$, $x = -2$

ZOD: Later

There is a horizontal tangent at $x = 0$ and $x = -2$

Write the equation of the tangent line and the normal line at the point given.

16. $f(x) = \dfrac{x-1}{x+1}$ at $x = 2$

Point

$f(2) = \dfrac{(2)-1}{(2)+1}$

$f(2) = \frac{1}{3}$

$(2, \frac{1}{3})$

Slope

$f' = \dfrac{(x-1)'(x+1) - (x-1)(x+1)'}{(x+1)^2}$

$= \dfrac{(1)(x+1) - (x-1)(1)}{(x+1)^2}$

$= \dfrac{x+1-x+1}{(x+1)^2}$

$f' = \dfrac{2}{(x+1)^2}$

tangent slope at $x=4$

$f'(4) = \dfrac{2}{[(4)+1]^2}$

$= \dfrac{2}{[5]^2}$

$f'(4) = \frac{2}{25}$

Normal slope at $x = 4$

$\perp m = \frac{25}{-2}$

tangent line

$y - y_1 = m(x - x_1)$

$y - (\frac{1}{3}) = \frac{2}{25}(x - (4))$

Normal line

$y - y_1 = m(x - x_1)$

$y - (\frac{1}{3}) = \frac{25}{-2}(x - (4))$

Find $f'(2)$ given the following.

$g(2) = 3$ and $g'(2) = -2$

$h(2) = -1$ and $h'(2) = 4$

17. $f(x) = 2g(x) + h(x)$

$f'(x) = 2g'(x) + h'(x)$

$f'(2) = 2g'(2) + h'(2)$

$= 2(-2) + (4)$

$= -4 + 4$

$f'(2) = 0$

18. $f(x) = 4 - h(x)$

$f'(x) = 4 - h'(x)$

$f'(2) = 4 - h'(2)$

$= 4 - 4$

$f'(2) = 0$

19. $f(x) = \dfrac{g(x)}{h(x)}$

$f'(x) = \dfrac{g'(x)h(x) - g(x)h'(x)}{h^2(x)}$

$f'(2) = \dfrac{g'(2)h(2) - g(2)h'(2)}{h^2(2)}$

$= \dfrac{(-2)(-1) - (3)(4)}{(-1)^2}$

$= \dfrac{2 - 12}{1}$

$= \frac{-10}{1}$

$f'(2) = -10$

20. $f(x) = g(x)h(x)$

$f'(x) = g'(x)h(x) + g(x)h'(x)$

$f'(2) = g'(2)h(2) + g(2)h'(2)$

$= (-2)(-1) + (3)(4)$

$= 2 + 12$

$f'(2) = 14$

MULTIPLE CHOICE

1. Suppose $f(x)$ is a differentiable function with $f(1) = 2, f(2) = -2, f'(2) = 5, f'(1) = 3,$ and $f(5) = 1$. An equation of a line tangent to the graph of f is

Point — Slope
$(1,2) \longrightarrow m=3 \Longrightarrow y-2=3(x-1)$
$(2,-2) \longrightarrow m=5 \Longrightarrow y+2=5(x-2)$
$(5,1) \longrightarrow m=?$

(A) $y - 3 = 2(x - 1)$
(B) $y - 2 = (x - 1)$
(C) $y - 3 = 5(x - 1)$
(D) $y - 2 = 3(x - 1)$
(E) $y - 1 = 5(x - 2)$

2. Let f and g be differentiable functions with the following properties:
   I. $f(x) < 0$ for all x
   II. $g(5) = 2$

If $h(x) = \frac{f(x)}{g(x)}$ and $h'(x) = \frac{f'(x)}{g(x)}$, then $g(x) =$

(A) $\frac{1}{f'(x)}$
(B) $f(x)$
(C) $-f(x)$
(D) 0
(E) 2

$h'(x) = \frac{f' \cdot g - f \cdot g'}{g^2} = \frac{f'(x)}{g(x)}$ but how?

g' must be a constant for $-f \cdot g'$ to disappear.
Since $g(5)=2$, then $g=2$.

$h'(x) = \frac{f' \cdot 2 - f \cdot (2)'}{(2)^2} = \frac{2f' - f \cdot (0)}{4} = \frac{2f'}{4} = \frac{f'}{2}$

3. At what point on the graph of $y = \frac{1}{2}x^2 - \frac{3}{2}$ is the tangent line parallel to the line $4x - 8y = 5$?

(A) $\left(\frac{1}{2}, -\frac{3}{8}\right)$
(B) $\left(\frac{1}{2}, -\frac{11}{8}\right)$
(C) $\left(2, \frac{3}{8}\right)$
(D) $\left(2, \frac{1}{2}\right)$
(E) $\left(-\frac{1}{2}, -\frac{11}{8}\right)$

1) Find slope of line
2) Find //m
3) Find y' of $y=\frac{1}{2}x^2-\frac{3}{2}$
4) Set $y' = //m$
5) Subst x-value into $y=\frac{1}{2}x^2-\frac{3}{2}$

① $\boxed{-8y=-4x+5 \quad y=\frac{1}{2}x-\frac{5}{8} \quad m=\frac{1}{2}}$

② $\boxed{//m=\frac{1}{2}}$

③ $\boxed{y=\frac{1}{2}x^2-\frac{3}{2} \quad y'=x}$

④ $\boxed{y'=//m \quad x=\frac{1}{2}}$

⑤ $y=\frac{1}{2}x^2-\frac{3}{2}$
$y(\frac{1}{2})=\frac{1}{2}(\frac{1}{2})^2-\frac{3}{2}$
$=\frac{1}{8}-\frac{12}{8}$
$y(\frac{1}{2})=-\frac{11}{8} \longrightarrow \therefore (\frac{1}{2}, -\frac{11}{8})$

4. If $f(x)$ is continuous and differentiable and $f(x) = \begin{cases} ax^4 + 5x; & x \le 2 \\ bx^2 - 3x; & x > 2 \end{cases}$, then find the value of b.

(A) 0.5
(B) 0
(C) 2
(D) 6
(E) There is no value of b.

Continuous At $x=2$
$a(2)^4+5(2)=b(2)^2-3(2)$
$16a+10=4b-6$
$16a+16=4b$
$4a+4=b$

Differentiable @ $x=2$
$(ax^4+5x)'=(bx^2-3x)'$
$4ax^3+5=2bx-3$
$4a(2)^3+5=2b(2)-3$
$4a(8)+5=4b-3$
$32a+5=4b-3$
$32a+8=4b$
$8a+2=b$

$4a+4=b$ — $8a+2=b$
$4a+4=8a+2$ — $8(\frac{1}{2})+2=b$
$2=4a$ — $4+2=b$
$\frac{1}{2}=a$ — $6=b$

You are allowed to use a graphing calculator for #5

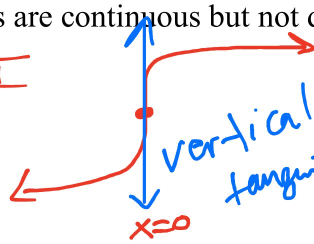
5. Which of the following functions are continuous but not differentiable at $x = 0$?

I. $f(x) = x^{\frac{1}{3}}$

II. $g(x) = |x|$

III. $h(x) = x|x|$

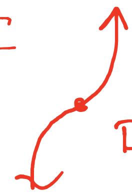
I



vertical tangent

$x=0$

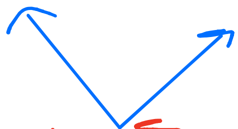
III



Differentiable at $x=0$

(A) I only
(B) II only
(C) I and II
(D) II and III
(E) I, II, and III

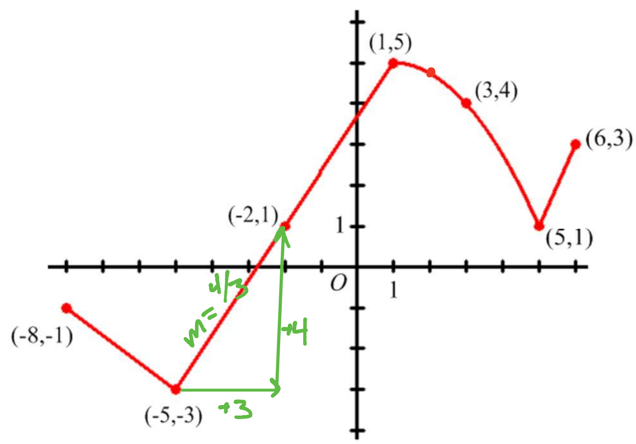
II



(0,0) corner point

FREE RESPONSE

Your score: ____ out of 4



Graph of g

1. A continuous function g is defined on the closed interval $-8 \le x \le 6$ and is shown above.

(a) Find the approximate value of $g'(4)$. Show the computations that lead to your answer.

What is the slope at 4?

Not part of answer

The average rate of change on [3,5] is a good estimate for the instantaneous rate of change at 4.

Approximate $g'(4) \approx ARC = \dfrac{g(b)-g(a)}{b-a}$

$= \dfrac{g(3)-g(5)}{3-5}$

$= \dfrac{(4)-(1)}{-2}$ +1

Approximate $g'(4) \approx \dfrac{3}{-2}$ +1

(b) Let h be the function defined by $h(x) = \dfrac{g(x)}{x^2+1}$. Find $h'(-2)$.

$h'(x) = \dfrac{g'(x)\cdot(x^2+1) - g(x)\cdot(x^2+1)'}{(x^2+1)^2}$

$h'(x) = \dfrac{g'(x)\cdot(x^2+1) - g(x)(2x)}{(x^2+1)^2}$ +1

$h'(-2) = \dfrac{g'(-2)[(-2)^2+1] - g(-2)[2(-2)]}{[(-2)^2+1]^2}$

$= \dfrac{(\frac{4}{3})[4+1] - (1)[-4]}{[4+1]^2}$

$= \dfrac{\frac{4}{3}(5)+4}{[5]^2}$

$= \dfrac{(\frac{20}{3}+4)}{25}\cdot\dfrac{3}{3}$

$= \dfrac{20+12}{75}$

$h'(-2) = \dfrac{32}{75}$ +1

Why does $g'(-2) = 2$?

Because g is linear at $x=-2$, when linear, $g' =$ slope of line g, which is $\frac{4}{3}$.