Practice FRQ 5.6

1. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

t (hours)	0	1	5	8	10
$G(t)$ (thousands of gallons)	22	18.5	19.5	22	24

A gasoline storage tank is filled by a pipeline from a refinery. At the same time, gasoline flows from the tank into trucks that will make deliveries to gasoline stations. The amount of gasoline in the storage tank at time t is given by the twice-differentiable function G, where t is measured in hours and $0 \le t \le 10$. Values of $G(t)$, in thousands of gallons, at selected times t are given in the table above. It is known that $G''(t) > 0$ for $5 \le t < 10$.

(a) Use the data in the table to estimate the rate of change of the amount of gasoline in the storage tank at time $t = 3$ hours. Show the computations that lead to your answer. Indicate units of measure.

Please respond on separate paper, following directions from your teacher.

(b) For $0 < t < 10$, is there a time t at which $G(t)$ is increasing at a rate of 0.2 thousand gallons per hour? Justify your answer.

Please respond on separate paper, following directions from your teacher.

(c) It is known that $G'(5) = 0.5$. Use the locally linear approximation for G at time $t = 5$ to approximate the amount of gasoline in the storage tank at time $t = 7$. Is this approximation an overestimate or an underestimate for the actual amount of gasoline in the storage tank at time $t = 7$? Give a reason for your answer.

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Please respond on separate paper, following directions from your teacher.

(d) The rate at which gasoline flows out of the storage tank into trucks at time t can be modeled by the function R defined by $R(t) = \frac{100t}{t^2+4}$, where t is measured in hours, and $R(t)$ is measured in thousands of gallons. Based on the model, at what time t, for $0 \leq t \leq 10$, is the rate at which gasoline flows out of the storage tank an absolute maximum? Justify your answer.

Please respond on separate paper, following directions from your teacher.

Part A

Substitution of function values is required. The answer does not need to be simplified.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1

The student response accurately includes a correct approximation with units.

Solution:

$G'(3) \approx \frac{G(5)-G(1)}{5-1} = \frac{19.5-18.5}{5-1} = \frac{1}{4} = 0.25$ thousand gallons per hour (which is equivalent to 250 gallons per hour)

Part B

The first point requires identification of the difference quotient. The substitution of function values and conclusion impacts the second point.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2

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The student response accurately includes both of the criteria below.

- ☐ $\frac{G(10)-G(0)}{10-0}$
- ☐ justification using Mean Value Theorem

Solution:

G is twice differentiable. $\Rightarrow$ G is differentiable. $\Rightarrow$ G is continuous.

$\frac{G(10)-G(0)}{10-0} = \frac{24-22}{10-0} = \frac{2}{10} = 0.2$ thousand gallons per hour (which is equivalent to 200 gallons per hour)

Therefore, by the Mean Value Theorem, there is a time t, for $0 < t < 10$, at which $G'(t) = 0.2$.

Part C

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2 ✓

The student response accurately includes both of the criteria below.

- ☐ approximation of $G(7)$
- ☐ underestimate with reason

Solution:

$G(7) \approx G(5) + G'(5)(7-5) = 19.5 + (0.5)(2) = 20.5$ thousand gallons

$G''(t) > 0$ for $5 \le t \le 7$, and so the graph of G is concave up for $5 \le t \le 7$. Therefore, the approximation for $G(7)$ using the locally linear approximation for G at time $t = 5$ is an underestimate for the actual amount.

Part D

Note: Sign charts are a useful tool to investigate and summarize the behavior of a function. By itself a sign chart for $f'(x)$ or $f''(x)$ is not a sufficient response for a justification.

The response is eligible for additional points based on consistent answers using an incorrect $R'(t)$ with a maximum of one computational error.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2	3	4

The student response accurately includes all four of the criteria below.

- ☐ $R'(t)$
- ☐ sets $R'(t) = 0$
- ☐ identifies $t = 2$ as a candidate
- ☐ answer with justification

Solution:

$$R'(t) = \frac{100(t^2+4) - 100t(2t)}{(t^2+4)^2} = \frac{400 - 100t^2}{(t^2+4)^2} = \frac{100(4-t^2)}{(t^2+4)^2} = 0$$
$$\Rightarrow 4 - t^2 = 0 \Rightarrow t = 2$$

Because $R'(t)$ changes from positive to negative at $t = 2$, R has a relative maximum at $t = 2$. R is increasing on $0 \le t \le 2$ and R is decreasing on $2 \le t \le 10$. Since $t = 2$ is the only critical point of R on the interval $0 \le t \le 10$ that is the location of a relative maximum, it is also the location of the absolute maximum of R on the interval $0 \le t \le 10$.

2. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

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Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

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t (hours)	0	5	15	30	35
$A(t)$ (gallons)	10	18	25	16	8

The number of gallons of olive oil in a tank at time t is given by the twice-differentiable function A, where t is measured in hours and $0 \le t \le 35$. Values of $A(t)$ at selected times t are given in the table above.

(a) Use the data in the table to estimate the rate at which the number of gallons of olive oil in the tank is changing at time $t = 10$ hours. Show the computations that lead to your answer. Indicate units of measure.

Please respond on separate paper, following directions from your teacher.

(b) For $0 \le t \le 30$, is there a time t at which $A'(t) = \frac{1}{5}$? Justify your answer.

Please respond on separate paper, following directions from your teacher.

(c) The number of gallons of olive oil in the tank at time t is also modeled by the function G defined by $G(t) = 5t - \frac{2}{3}(t+9)^{\frac{3}{2}} + 28$, where t is measured in hours and $0 \le t \le 35$. Based on the model, at what time t, for $0 \le t \le 35$, is the number of gallons of olive oil in the tank an absolute maximum? Justify your answer.

Please respond on separate paper, following directions from your teacher.

(d) For the function G defined in part (c), the locally linear approximation near $t = 7$ is used to approximate $G(8)$. Is this approximation an overestimate or an underestimate for the value of $G(8)$? Give a reason for your answer.

Please respond on separate paper, following directions from your teacher.

Part A

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Substitution of function values is required. The answer does not need to be simplified.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1 ✓

The student response accurately includes the criteria below.

- ☐ approximation with units

Solution:

$A'(10) \approx \frac{A(15)-A(5)}{15-5} = \frac{25-18}{15-5} = \frac{7}{5}$ gallons per hour

Part B

The first point requires identification of the difference quotient. The substitution of function values and conclusion impacts the second point.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2 ✓

The student response accurately includes both of the criteria below.

- ☐ $\frac{A(30)-A(0)}{30-0}$
- ☐ justification using Mean Value Theorem

Solution:

A is twice differentiable. $\Rightarrow$ A is differentiable. $\Rightarrow$ A is continuous.

$\frac{A(30)-A(0)}{30-0} = \frac{16-10}{30-0} = \frac{6}{30} = \frac{1}{5}$

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Therefore, by the Mean Value Theorem, there is a time t, for $0 \le t \le 30$, at which $A'(t) = \frac{1}{5}$.

Part C

Note: Sign charts are a useful tool to investigate and summarize the behavior of a function. By itself a sign chart for $f'(x)$ or $f''(x)$ is not a sufficient response for a justification.

The response is eligible for additional points based on consistent answers using an incorrect $G'(t)$ with a maximum of one computational error.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2	3	4

The student response accurately includes all four of the criteria below.

- ☐ $G'(t)$
- ☐ sets $G'(t) = 0$
- ☐ identifies $t = 16$ as a candidate
- ☐ answer with justification

Solution:

$$G'(t) = 5 - (t+9)^{\frac{1}{2}} = 0 \Rightarrow (t+9)^{\frac{1}{2}} = 5$$
$$\Rightarrow t + 9 = 25 \Rightarrow t = 16$$

Because $G'(t)$ changes from positive to negative at $t = 16$, G has a relative maximum at $t = 16$.

Since $t = 16$ is the only critical point of G on the interval $0 \le t \le 35$ that is the location of a relative maximum, it is also the location of the absolute maximum of G on the interval $0 \le t \le 35$.

Part D

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

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0	1	2

The student response accurately includes both of the criteria below.

- ☐ $G'' \ (t)$
- ☐ overestimate with reason

Solution:

$G'' \ (t) = -\frac{1}{2}(t+9)^{-\frac{1}{2}}$

$G'' \ (t) < 0$ for $7 \le t \le 8$, and so the graph of G is concave down for $7 \le t \le 8$. Therefore, the approximation of $G \ (8)$ using the locally linear approximation of G near $t = 7$ is an overestimate.