Avon High School
AP Calculus AB

SOLUTIONS

Period _____ Score ______ / 10

Skill Builder: Topic 5.10 – Introduction to Optimization Problems
Topic 5.11 – Solving Optimization Problems

Solve each of the following problems. Be sure to show all necessary work and justify why the maximum or minimum occurs.

1.) Find two numbers whose sum is 10 for which the sum of their squares is a minimum.

Primary Equation: $S = x^2 + y^2$ **Secondary Equation**: $10 = x + y$

$S = x^2 + (10-x)^2$

$S' = 2x - 2(10-x)$

$\underline{S' = 0}$

$4x - 20 = 0$

$x = \frac{20}{4} = 5$

$S'' = 4 > 0$ (we obtain a minimum)

The numbers are 5 and 5.

2.) Find nonnegative numbers x and y whose sum is 75 and for which the value xy^2 is as large as possible.

Primary Equation: $P = xy^2$ **Secondary Equation**: $75 = x + y$

$P = x(75-x)^2$

$P' = 1(75-x)^2 + x \cdot 2(75-x)(-1)$

$P' = (75-x)[75 - x - 2x]$

$P' = (75-x)(75-3x)$

$\underline{P' = 0}$

$(75-x)(75-3x) = 0$

$x = 25,\ 75$

$P'' = (-x)(75-3x) + (75-x)(-3)$

$P'' = -75x + 3x^2 - 225 + 3x$

$P'' = 3x^2 - 72x - 225$

$P''(25) = 3(25)^2 - 72(25) - 225 < 0$ a max

$P''(75) = 3(75)^2 - 72(75) - 225 > 0$ a min

The numbers are 25 and 50.

3.) After winning a U.S. Open Tennis match at Arthur Ashe Stadium, players fire several tennis balls into the stands as souvenirs. Some players try to hit the ball out of the stadium, at a height of 518 feet. It has never been done. If the ball is propelled from a tennis racket straight up, the ball's height after t seconds is given by $h = v_0 t - 16t^2$ where v_0 is the initial velocity. What is v_0 in order for the ball to reach a maximum height of 518 feet? How long will it take for the ball to reach that height?

Primary Equation: $h = v_0 t - 16t^2$ **Secondary Equation**: $518 = v_0 t - 16t^2$

$v_0 = 32\left(\frac{\sqrt{518}}{4}\right) = 8\sqrt{518}$ ft/sec

$h' = v_0 - 32t$

$\underline{h' = 0}$

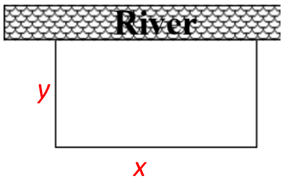
$v_0 = 32t$

$518 = 32t^2 - 16t^2$

$518 = 16t^2$

$t = \frac{\sqrt{518}}{4} \approx 5.689$ seconds

4.) A farmer has 2,000 feet of fencing to enclose a pasture area. The field will be in the shape of a rectangle and will be placed against a river where there is no fencing needed. What is the largest area field that can be created and what are its dimensions?



Primary Equation: $A = xy$ **Secondary Equation**: $2000 = x + 2y$

$A = x\left(\frac{2000 - x}{2}\right)$ $\underline{A' = 0}$ $2y = 2000 - x \rightarrow y = \frac{2000 - x}{2}$

$A = 1000x - \frac{1}{2}x^2$ $1000 - x = 0$ $A'' = -1 < 0$ a max

$A' = 1000 - x$ $x = 1000$ $y(1000) = \frac{2000 - 1000}{2} = 500$

The dimensions of the area should be 1000 ft by 500 ft with an area of $500000\,\text{ft}^2$.

5.) A fisheries biologist is stocking fish in a lake. She knows that when there are *n* fish per unit of water, the average weight of each fish will be $W(n) = 500 - 2n$, measured in grams. What is the value of *n* that will maximize the total fish weight after one season. Complete the chart.

n	0	1	10	50	200
$W(n)$	500	498	480	400	100
Weight of fish	0	498	4800	20000	20000

Primary Equation: $T_{weight} = n(W)$ **Secondary Equation**: $W = 500 - 2n$

$T = 500n - 2n^2 \rightarrow T' = 500 - 4n$ $\underline{T' = 0}$ $T'' = -4 < 0$ a max

$500 - 4n = 0$

$n = 125$

The total weight of the fish will be a maximum when there are 125 fish.

$(T_{weight} = 500(125) - 2(125)^2 = 31250)$

6.) The size of a population of bacteria introduced to a food grows according to the formula $P(t) = \frac{6000t}{60 + t^2}$ where *t* is measured in weeks. Determine when the bacteria will reach its maximum size. What is the maximum size of the population?

Primary Equation: $P(t) = \frac{6000t}{60 + t^2}$ **Secondary Equation**: not necessary

$$P'(t) = \frac{6000(60 + t^2) - 6000t(2t)}{(60 + t^2)^2} = \frac{360000 + 6000t^2 - 12000t^2}{(60 + t^2)^2} = \frac{-6000\left(t^2 - 60\right)}{(60 + t^2)^2}$$

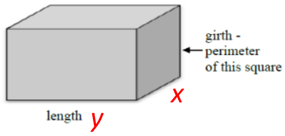
$P'(t) = 0$ when $t^2 - 60 \rightarrow t = \sqrt{60}$ (There cannot be a negative number of bacteria)

$P'(t)$ is undefined at no value of t.

$P'(1) = \frac{-6000(1 - 60)}{\left((60 + 1)^2\right)} > 0$ $P'(\sqrt{61}) = \frac{-6000(61 - 60)}{\left((60 + 61^2)^2\right)} < 0$

The bacteria in the food will reach its maximum size at $t = \sqrt{60} \approx 7.745$ weeks.

7.) The U.S. Postal Service will accept a box for a domestic shipping only if the sum of the length and the girth (distance around) does not exceed 108 inches. Find the dimensions of the largest volume box with a square end that can be sent.



Primary Equation: $V = x^2 \cdot y$ **Secondary Equation**: $108 = 4x + y$

$V = x^2(108 - 4x) = 108x^2 - 4x^3$

$V' = 216x - 12x^2 = 12x(18 - x)$ $V' = 0$ when $12x(18 - x) \rightarrow x = \not{0}, 18$

$V'' = 216 - 24x$ $V''(18) = 216 - 24(18) < 0$ a max

$y(18) = 108 - 4(18) = 36$

The dimensions of the largest accepted box would be 18 inches by 36 inches.

8.) Blood pressure in a patient will drop by an amount $D(x)$ where $D(x) = 0.025x^2(30 - x)$ where x is the amount of drug injected in cm^3. Find the dosage that provides the greatest drop in blood pressure. What is the drop in blood pressure?

Primary Equation: $D(x) = 0.025x^2(30 - x)$ **Secondary Equation:** not necessary

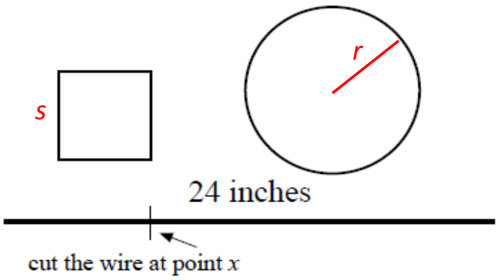
$D(x) = 0.025x^2(30 - x) = 0.75x^2 - 0.025x^3$

$D'(x) = 1.5x - 0.075x^2 = 0.075x(20 - x) = 0 \Rightarrow x = 0, 20$

$D''(x) = 1.5 - 0.15x \Rightarrow D''(20) = -1.5 < 0 \Rightarrow$ max at $x = 20$ $D(20) = 0.025(20)^2(10) = 100$

The greatest drop in blood pressure is 100 when the dosage is 20 cm^3.

9.) A wire 24 inches long is cut into two pieces as shown by the figure below. One piece is to be shaped into a square and the other piece into a circle. Let x be the point where the cut is made. Assume that the square uses the left piece and the circle the right piece. Complete the chart and find the location of the cut where the total area enclosed by the square and the circle is a maximum.



x	4	8	12	20	x
Area square	1^2	2^2	3^2	5^2	$\left(\frac{x}{4}\right)^2$
Area circle	$\left(\frac{20}{2p}\right)^2 p$	$\left(\frac{16}{2p}\right)^2 p$	$\left(\frac{12}{2p}\right)^2 p$	$\left(\frac{4}{2p}\right)^2 p$	$\left(\frac{24-x}{2p}\right)^2 p$
Total area	$1^2+\left(\frac{20}{2p}\right)^2 p$	$2^2+\left(\frac{16}{2p}\right)^2 p$	$3^2+\left(\frac{12}{2p}\right)^2 p$	$5^2+\left(\frac{12}{2p}\right)^2 p$	$\left(\frac{x}{4}\right)^2+\frac{(24-x)^2}{4p}$

Primary equation: $A = s^2 + pr^2$ **Secondary Equation:** $s = \frac{x}{4}$ $r = \frac{(24-x)}{2p}$

$$A(x) = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{(24-x)}{2\pi}\right)^2 = \left(\frac{x}{4}\right)^2 + \frac{(24-x)^2}{4\pi}$$

$$A'(x) = 2\left(\frac{x}{4}\right)\left(\frac{1}{4}\right) + \frac{2(24-x)(-1)}{4\pi}$$

$$A'(x) = \left(\frac{x}{8}\right) + \frac{(x-24)}{2\pi}$$

$$A''(x) = \frac{1}{8} + \frac{1}{2\pi} > 0 \Rightarrow \text{ min at any critical value of } x.$$

$$A'(x) = \left(\frac{x}{8}\right) + \frac{(x-24)}{2\pi} = 0$$

$$\frac{(x-24)}{2\pi} = -\frac{x}{8}$$

$$8x - 192 = -2\pi x$$

$$8x + 2\pi x = 192$$

$$x = \frac{192}{8+2\pi} \text{ min}$$

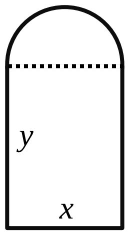
maximum is at one endpoint $x = 0$ (just make a circle) or $x = 24$ (just make a square)

$$A(0) = \left(\frac{0}{4}\right)^2 + \frac{(24-0)^2}{4\pi} = \frac{24^2}{4\pi} = 45.8366\ldots in^2$$

$$A(24) = \left(\frac{24}{4}\right)^2 + \frac{(24-24)^2}{4\pi} = 36\ in^2$$

Maximum area of $\frac{144}{\pi}$ in^2 occurs when you use the entire 24 inches to make a circle of radius $\frac{12}{\pi}$ in.

10.) A designer of custom windows wishes to build a Norman Window with a total outside perimeter of 60 feet. How should the window be designed to maximize the area of the window? A Norman Window contains a rectangle bordered above by a semicircle.



Primary equation: $A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$ **Secondary Equation:** $P = 2y + x + \frac{1}{2}\pi x$

$$P = 60 \Rightarrow 2y + x + \frac{1}{2}\pi x = 60$$

$$y = \frac{60 - x - \frac{1}{2}\pi x}{2} = \frac{120 - 2x - \pi x}{4} = \frac{120 - (2+\pi)x}{4}$$

$$A(x) = x\left(\frac{120 - (2+\pi)x}{4}\right) + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = \frac{120x - (2+\pi)x^2}{4} + \frac{1}{2}\pi\left(\frac{x^2}{4}\right)$$

$$= \frac{1}{8}\left(240x - 2(2+\pi)x^2 + \pi x^2\right) = \frac{1}{8}\left(240x - (4+\pi)x^2\right)$$

$$A'(x) = \frac{1}{8}\left(240 - 2(4+\pi)x\right) = 0$$

$$\left(240 - 2(4+\pi)x\right) = 0$$

$$x_c = \frac{120}{4+\pi} = 16.8029\ldots$$

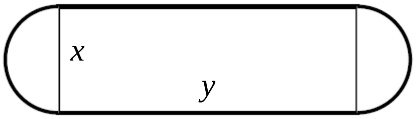
$$A''(x) = \frac{1}{8}\left(-2(4+\pi)\right) < 0 \Rightarrow \text{max}$$

$$y = \frac{120 - (2+\pi)\frac{120}{4+\pi}}{4} = \frac{120(4+\pi) - 120(2+\pi)}{4(4+\pi)}$$

$$= \frac{480 + 120\pi - 240 - 120\pi}{4(4+\pi)} = \frac{240}{4(4+\pi)} = \frac{60}{4+\pi} = \frac{x_c}{2}$$

The window should have rectangle with base 16.8029… ft and height 8.4014…ft and semicircular top with radius 8.4014…ft to have a maximum area of 252.0446…ft^2.

11.) A physical fitness area consists of a rectangular region with a semi-circle on each end as shown by the figure below. If the perimeter of the room is to be a 200-meter running track, find the dimensions that will make the area of the rectangular region as large as possible.



Primary equation:
$A = xy$

Secondary Equation:
$P = 2x + 2y + px$

$$P = 200 \Rightarrow 2y + \pi x = 200$$

$$y = \frac{200 - \pi x}{2}$$

$$A(x) = x\left(\frac{200 - \pi x}{2}\right) = \frac{1}{2}\left(200x - \pi x^2\right)$$

$$A'(x) = \frac{1}{2}(200 - 2\pi x)$$

$$y = \frac{200 - \pi\left(\frac{100}{\pi}\right)}{2} = 50$$

$$A'(x) = \frac{1}{2}(200 - 2\pi x) = 0$$

$$2\pi x = 200$$

$$x = \frac{100}{\pi}$$

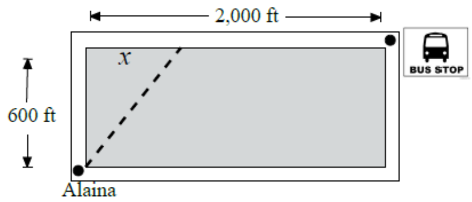
$$A''(x) = \frac{1}{2}(-2\pi) < 0$$

max at $x_{max} = \frac{100}{\pi}$

The maximum area will occur when the rectangle is

$\frac{100}{\pi}$ meters wide and 50 meters long.

12.) Alaina wants to get to the bus stop as quickly as possible. The bus stop is across a grassy park, 2,000 feet east and 600 feet north of her starting position. Alaina can walk along the edge of the park on the sidewalk at a rate of 6 feet/sec. She can also travel through the grass in the park, but only at a rate of 4 feet/sec (dogs are walked there, so she must move with care or get a surprise on her shoes). What path will get her to the bus stop the fastest?



Primary equation:

$$T = \frac{D_{grass}}{4} + \frac{D_{sidewalk}}{6}$$

Secondary Equation:

$$D_{grass} = \sqrt{x^2 + 600^2} \qquad D_{sidwalk} = 2000 - x$$

$$T = \frac{\sqrt{x^2+600^2}}{4} + \frac{2000-x}{6}$$

$$T'(x) = \frac{2x}{4\left(2\sqrt{x^2+600^2}\right)} - \frac{1}{6} = \frac{x}{4\left(\sqrt{x^2+600^2}\right)} - \frac{1}{6}$$

$$T'(x) = 0 \Rightarrow \frac{x}{4\left(\sqrt{x^2+600^2}\right)} = \frac{1}{6}$$

$$6x = 4\sqrt{x^2+600^2}$$

$$36x^2 = 16\left(x^2+600^2\right)$$

$$20x^2 = 16\left(600^2\right)$$

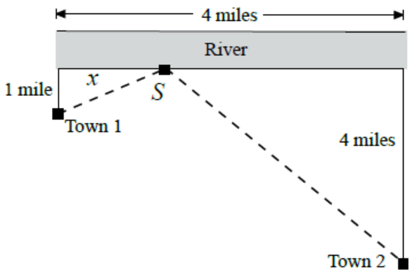
$$x_c = \sqrt{\frac{16\left(600^2\right)}{20}} = 600(2)\sqrt{\frac{1}{5}}$$

$$= \frac{1200\sqrt{5}}{5} = 240\sqrt{5} = 536.6563\ldots$$

$$T'(0) = -\frac{1}{6} < 0 \qquad T'(600) = \frac{600}{4\left(\sqrt{600^2+600^2}\right)} - \frac{1}{6} = \frac{1}{4\sqrt{2}} - \frac{1}{6} > 0 \text{ since } \sqrt{2} < 1.5 \Rightarrow 4\sqrt{2} < 6$$

$x = 240\sqrt{5}$ gives the least amout of time. She should walk across the grass to a point $240\sqrt{5} = 536.6563\ldots$ feet east of the corner of the park then east along the sidewalk to the bus stop. It will take about 445.1367…seconds or about 7.4189…minutes.

13.) On the same side of a straight river are two towns, and the townspeople want to build a pumping station, **S**, that supplies water to them. The pumping station is to be at the river's edge with pipes extending straight to the two towns. The distances are shown in the figure below. Where should the pumping station be located to minimize the total length of pipe?
This problem poses a challenging derivative, but you can do it!



Primary equation: $L = D_1 + D_2$

Secondary Equation: $D_1 = \sqrt{x^2 + 1^2}$ $\quad D_2 = \sqrt{(4-x)^2 + 4^2}$

$$L(x) = \sqrt{x^2+1} + \sqrt{(4-x)^2+16} = \sqrt{x^2+1} + \sqrt{32-8x+x^2}$$

$$L'(x) = \frac{2x}{2\sqrt{x^2+1}} + \frac{2x-8}{2\sqrt{32-8x+x^2}}$$

$$L'(x) = \frac{x}{\sqrt{x^2+1}} + \frac{x-4}{\sqrt{32-8x+x^2}} = 0$$

$$\frac{x}{\sqrt{x^2+1}} = -\frac{x-4}{\sqrt{32-8x+x^2}}$$

$$x\sqrt{32-8x+x^2} = -(x-4)\sqrt{x^2+1}$$

$$x^2(32-8x+x^2) = (x^2-8x+16)(x^2+1)$$

$$32x^2 - 8x^3 + x^4 = x^4 - 8x^3 + 17x^2 - 8x + 16$$

$$15x^2 + 8x - 16 = 0$$

$$x = \frac{-8 \pm \sqrt{64 - 4(15)(-16)}}{30} = \frac{-8 \pm \sqrt{1024}}{30} = \frac{-8 \pm 32}{30}$$

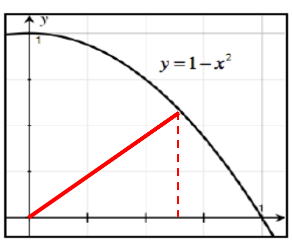
$$= -\frac{40}{30}, \frac{24}{30} = -\frac{4}{3}, \frac{4}{5}$$

$x = \frac{4}{5}$ is the only possible critcal value. $L'(0) = \frac{-4}{\sqrt{32}} < 0$ $\qquad L'(4) = \frac{4}{\sqrt{17}} > 0$

$x = \frac{4}{5}$ produces a minimum because $L'(x)$ changes from positive to negative at $x = \frac{4}{5}$.

The pumping station should be located on the river $\frac{4}{5}$ miles up the river toward Town 2.

14.) Below is the graph of $y = 1 - x^2$ for $x \geq 0$. Find the point on this curve which is closest to the origin.

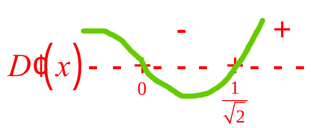


Primary equation: $D = \sqrt{x^2 + y^2}$

Secondary Equation: $y = 1 - x^2$

$$D(x) = \sqrt{x^2 + (1-x^2)^2} = \sqrt{x^2 + (1 - 2x^2 + x^4)} = \sqrt{x^4 - x^2 + 1}$$

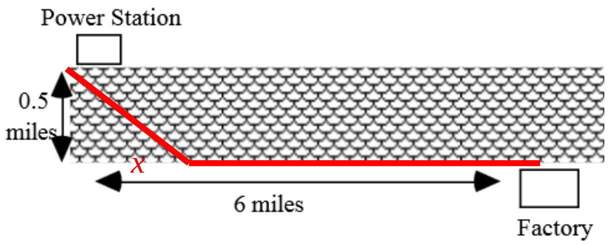
$$D'(x) = \frac{4x^3 - 2x}{2\sqrt{x^4 - x^2 + 1}} = \frac{2x^3 - x}{\sqrt{x^4 - x^2 + 1}} = 0 \Rightarrow x(2x^2 - 1) = 0 \Rightarrow x = 0, \frac{1}{\sqrt{2}}$$



numerator is an open up cubic

The point is closest when $x = \frac{1}{\sqrt{2}}$ because $D'(x)$ changes signs from negative to positive.

15.) A power station is on one side of a river that is 0.5 miles wide, and a factory is 6 miles downstream on the other side. It costs $6,000 per mile to run power lines overhead and $8,000 per mile to run them underwater. Find the most economical path to lay transmission lines from the station to the factory.



Primary equation: $C = 8000D_{water} + 6000D_{overhead}$

Secondary Equation: $D_w = \sqrt{x^2 + 0.5^2}$ $\quad D_{oh} = 6 - x$

$C(x) = 8000\sqrt{x^2 + 0.25} + 6000(6 - x)$

$C'(x) = \dfrac{8000(2x)}{2\sqrt{x^2 + 0.25}} - 6000 = 0$

$\dfrac{8000(x) - 6000\sqrt{x^2 + 0.25}}{\sqrt{x^2 + 0.25}} = 0$ $\qquad C'(0) = \dfrac{-6000\sqrt{0.25}}{\sqrt{0.25}} < 0$ $\qquad C'(2) = \dfrac{16000 - 6000\sqrt{5}}{\sqrt{5}} > 0$

$2000\left(4x - 3\sqrt{x^2 + 0.25}\right) = 0$

$4x = 3\sqrt{x^2 + 0.25}$

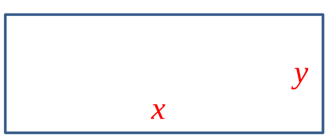
$16x^2 = 9\left(x^2 + 0.25\right) = 9x^2 + 2.25$

$7x^2 = 2.25 \Rightarrow x_c = \sqrt{\dfrac{2.25}{7}} = 0.5669\ldots$

There is a minimum cost when $x = \sqrt{\dfrac{2.25}{7}} = 0.5669\ldots$ because $C'(x)$ changes from negative to positive.

The most economical path will run the power line underwater across the river to a point $\sqrt{\dfrac{2.25}{7}} = 0.5669\ldots$ miles toward the factory then overhead to factory.

16.) A rectangular area is to be fenced in using two types of fencing. The front and back uses fencing costing $5 a foot while the sides use fencing costing $4 a foot. If the area of the rectangle must contain 500 square feet, what should be the dimensions of the rectangle in order to keep the cost to a minimum?



Primary equation: $C = C_{fb} + C_{sides}$

Secondary equation: $xy = 500$

$C(x) = 5(2x) + 4(2y) = 10x + 8y = 10x + 8\left(\dfrac{500}{x}\right) = 10x + \dfrac{4000}{x}$

$C'(x) = 10 + \left(-\dfrac{4000}{x^2}\right) = \dfrac{10x^2 - 4000}{x^2}$ $\qquad C''(x) = \dfrac{8000}{x^3} > 0 \Rightarrow \min$

$C'(x) = 0 \Rightarrow 10x^2 - 4000 = 0$ $\qquad x = 20 \Rightarrow y = \dfrac{500}{20} = 25$

$x = \sqrt{\dfrac{4000}{10}} = \sqrt{400} = 20$ $\qquad C = 10(20) + 8(25) = 400$

The minimum cost of fencing, $400, occurs when the rectangle is 20 feet along the front and back and 25 feet on the sides.

17.) The same rectangle area is to be build as in Problem 16, but now the builder hs $800 to spend. What is the largest area that can be fenced in using the same two types of fencing mention above.

Primary equation: $A = xy$ **Secondary equation:** $C = 10x + 8y = 800$

$$A(x) = xy = x\left(\frac{400-5x}{4}\right) = \frac{400x-5x^2}{4} \qquad 10x+8y=800 \Rightarrow y = \frac{800-10x}{8} = \frac{400-5x}{4}$$

$$A'(x) = \frac{400-10x}{4} = 0 \Rightarrow x = 40 \qquad A(40) = \frac{400(40)-5(40)^2}{4} = 2000$$

$$A''(x) = \frac{-10}{4} < 0 \Rightarrow \text{max}$$

The maximum area is 2000 ft^2 when the rectangle is 40 ft across the front and 50 ft on the sides.