Skill Builder: Topic 5.1 – Using the Mean Value Theorem

For the exercises below, determine whether Rolle's Theorem can be applied to the function in the indicated interval. If Rolle's Theorem can be applied, find all values of *c* that satisfy Rolle's Theorem.

1.) $f(x) = x^2 - 4x$ on $[0,4]$

$f(x)$ is continuous on $[0,4]$

$f(x)$ is differentiable on $(0,4)$

$f(0) = 0^2 - 4(0) = f(4) = 4^2 - 4(4) = 0$

Rolle's Theorem applies.

$f'(c) = 2c - 4$

$2c - 4 = 0$

$c = 2$

2.) $f(x) = x^2 - 11x + 30$ on $[5,6]$

$f(x)$ is continuous on $[5,6]$

$f(x)$ is differentiable on $(5,6)$

$f(5) = 5^2 - 11(5) + 30 = f(6) = 6^2 - 11(6) + 30 = 0$

Rolle's Theorem applies.

$f'(c) = 2c - 11$

$2x - 11 = 0$

$c = \frac{11}{2}$

3.) $f(x) = 4 - |x - 2|$ on $[-2,2]$

$f(x)$ is continuous on $[-2,2]$

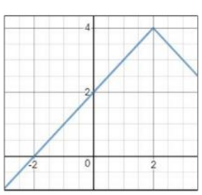
$f(x)$ is differentiable on $(-2,2)$

(Note: $f(x)$ is not differentiable at $x = 2$)

However, Rolle's Theorem only requires the function to be differentiable on the OPEN inteval.

$f(-2) = 4 - |-2 - 2| = 0 \neq f(2) = 4 - |2 - 2| = 4$

Rolle's Theorem does not apply.



4.) $f(x) = \sin x$ on $[0, 2\pi]$

$f(x)$ is continuous on $[0, 2\pi]$

$f(x)$ is differentiable on $(0, 2\pi)$

$f(0) = \sin(0) = 0 = f(2\pi) = \sin(2\pi) = 0$

Rolle's Theorem applies.

$f'(c) = \cos c$

$\cos c = 0$

$c = \frac{\pi}{2}, \frac{3\pi}{2}$

5.) $f(x) = \cos 2x$ on $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$

$f(x)$ is continuous on $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$

$f(x)$ is differentiable on $\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$

$f\left(\frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} = f\left(\frac{2\pi}{3}\right) = \cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$

Rolle's Theorem applies.

$f'(c) = -2\sin(2c)$

$-2\sin(2c) = 0$

$\sin(2c) = 0$

$2c = 0, \pi, 2\pi, \ldots$

$c = \frac{\pi}{2}$

Note: $\frac{\pi}{2}$ is the only result that lies within $\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$

6.) $f(x) = \frac{6x}{\pi} - 4\sin^2 x$ on $\left[0, \frac{\pi}{6}\right]$

$f(x)$ is continuous on $\left[0, \frac{\pi}{6}\right]$

$f(x)$ is differentiable on $\left(0, \frac{\pi}{6}\right)$

$f(0) = \frac{6(0)}{\pi} - 4\sin^2(0) = 0$

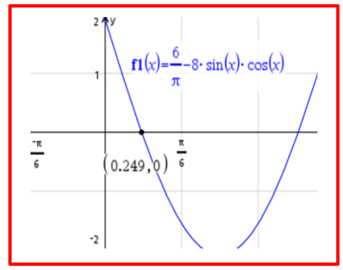
$f\left(\frac{\pi}{6}\right) = \frac{6\left(\frac{\pi}{6}\right)}{\pi} - 4\sin^2\left(\frac{\pi}{6}\right) = 1 - 4\left(\frac{1}{2}\right)^2 = 0$

Rolle's Theorem applies.

$f'(c) = \frac{6}{\pi} - 8\sin c \cdot \cos c$

$\frac{6}{\pi} - 8\sin c \cdot \cos c = 0$

$c = 0.249$



For the exercises below, apply the Mean Value Theorem to $f(x)$ on the indicated interval. Find all values of c which satisfy the Mean Value Theorem.

7.) $f(x) = x^2$ on $[-1, 2]$

$f(x)$ is continuous on $[-1, 2]$

$f(x)$ is differentiable on $(-1, 2)$

Mean Value Theorem applies.

$f'(c) = 2c$

$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{2^2 - (-1)^2}{3} = \frac{3}{3} = 1$

$2c = 1$

$c = \frac{1}{2}$

8.) $f(x) = x^3 - x^2 - 2x$ on $[-1, 1]$

$f(x)$ is continuous on $[-1, 1]$

$f(x)$ is differentiable on $(-1, 1)$

Mean Value Theorem applies.

$f'(c) = 3c^2 - 2c - 2$

$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{1^3 - 1^2 - 2(1) - ((-1)^3 - (-1)^2 - 2(-1))}{2}$

$= \frac{-2}{2} = -1$

$3c^2 - 2c - 2 = -1$

$3c^2 - 2c - 1 = 0$

$(3c + 1)(c - 1) = 0$

$c = -\frac{1}{3}$

Note: We disregard $c = 1$ because it does not occur on the interval $(-1, 1)$

9.) $f(x)=\dfrac{x+2}{x}$ on $\left[\dfrac{1}{2},2\right]$

$f(x)$ is continuous on $\left[\dfrac{1}{2},2\right]$

$f(x)$ is differentiable on $\left(\dfrac{1}{2},2\right)$

Mean Value Theorem applies.

$$f'(c)=-\frac{2}{c^2}$$

$$\frac{f(2)-f\left(\frac{1}{2}\right)}{2-\left(\frac{1}{2}\right)}=\frac{\frac{2+2}{2}-\frac{\frac{1}{2}+2}{\frac{1}{2}}}{\frac{3}{2}}=\frac{2-5}{\frac{3}{2}}=-2$$

$$-\frac{2}{c^2}=-2$$

$$-2=-2c^2$$

$$c^2=1$$

$$c=1$$

Note: We disregard $c=-1$ because it does not occur on the interval $\left(\dfrac{1}{2},2\right)$

10.) $f(x)=\sqrt{x-3}$ on $[3,7]$

$f(x)$ is continuous on $[3,7]$

$f(x)$ is differentiable on $(3,7)$

Mean Value Theorem applies.

$$f'(c)=\frac{1}{2}(c-3)^{-1/2}=\frac{1}{2\sqrt{c-3}}$$

$$\frac{f(7)-f(3)}{7-3}=\frac{\sqrt{7-3}-\sqrt{3-3}}{4}=\frac{\sqrt{4}-0}{4}=\frac{1}{2}$$

$$\frac{1}{2\sqrt{c-3}}=\frac{1}{2}$$

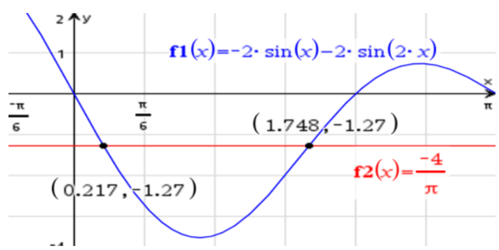
$$2\sqrt{c-3}=2$$

$$\sqrt{c-3}=1$$

$$c-3=1$$

$$c=4$$

Graph for Problem 14



11.) $f(x)=x^3$ on $[0,1]$

$f(x)$ is continuous on $[0,1]$

$f(x)$ is differentiable on $(0,1)$

Mean Value Theorem applies.

$$f'(c)=3c^2$$

$$\frac{f(1)-f(0)}{1-0}=\frac{1^3-0^3}{1}=1$$

$$3c^2=1$$

$$c^2=\frac{1}{3}$$

$$c=\sqrt{\frac{1}{3}}$$

Note: We disregard $c=-\sqrt{\dfrac{1}{3}}$ because it does not occur on the interval $(0,1)$

12.) $f(x)=2\cos x+\cos 2x$ on $[0,\pi]$

$f(x)$ is continuous on $[0,\pi]$

$f(x)$ is differentiable on $(0,\pi)$

The Mean Value Theorem applies.

$$f'(c)=-2\sin c-2\sin(2c)$$

$$\frac{f(\pi)-f(0)}{\pi-0}=\frac{2\cos\pi+\cos(2\pi)-(2\cos 0+\cos(2\cdot 0))}{\pi}$$

$$=\frac{-2+1-(2+1)}{\pi}=-\frac{4}{\pi}$$

$$-2\sin c-2\sin(2c)=-\frac{4}{\pi}$$

$$c=0.217,\ 1.748$$

13.) A trucker handed a ticket at a toll booth showing that in 2 hours the truck had covered 159 miles on a toll road in which the speed limit was 65 mph. The trucker was cited for speeding. Why?

Average rate of change (average velocity) = $\dfrac{\Delta \text{ distance}}{\Delta \text{ time}} = \dfrac{159\,\text{mi}}{2\,\text{hr}} = 79.5\,\text{mph}$

The trucker could be cited for speeding because his average velocity was 79.5 mph on a stretch of road which had a speed limit of 65 mph.

14.) A marathoner ran the 26.2 mile New York City Marathon in 2 hours, 12 minutes. Like all the other runners, he started from a standing position. During the last 5 meters, his leg cramped and he fell down and had to roll across the finish line. Prove that at least twice, the marathoner was running at exactly 11 mph.

Average rate of change (average velocity) = $\dfrac{\Delta \text{ distance}}{\Delta \text{ time}} = \dfrac{26.2\,\text{mi}}{2.2\,\text{hr}} \approx 11.910\,\text{mph}$

The runner had an average velocity of approximately 11.9 mph. Because he started from a stationary position and had to crawl across the finish line, there were two moments where her velocity had to be 11 mph.

15.) The order and transportation cost C of bottles of Pepsi® is approximated by the function:

$C(x) = 10{,}000\left(\dfrac{1}{x} + \dfrac{x}{x+3}\right)$ where x is the order size of bottles of Pepsi® in hundreds.

According to Rolle's Theorem, the rate of change of cost must be zero for some order size in the interval $[3,6]$. Find the order size.

The order size is approximately 409 (or 410) bottles of Pepsi.

This problem can be solved completely by using a graphing calculator. For students with a non-CAS calculator, you will have to graph the derivative of $C(x)$ and determine where it crosses the x-axis.

$c(x) := 10000 \cdot \left(\dfrac{1}{x} + \dfrac{x}{x+3}\right)$ *Done*

$\dfrac{d}{dx}(c(x))$ $\quad \dfrac{30000}{(x+3)^2} - \dfrac{10000}{x^2}$

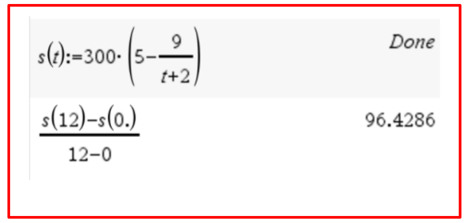
$\text{solve}\left(\dfrac{30000}{(x+3)^2} - \dfrac{10000}{x^2} = 0., x\right)$

$x = -1.09808$ or $x = 4.09808$

16.) A car company introduces a new car for which the number of cars sold S is the function $S(t) = 300\left(5 - \frac{9}{t+2}\right)$ where t is the time in months.

a.) Find the average rate of cars sold over the first 12 months.

$$\frac{S(12) - S(0)}{12 - 0} = \frac{300\left(5 - \frac{9}{12+2}\right) - \left(300\left(5 - \frac{9}{0+2}\right)\right)}{12} \approx 96.428$$



The average rate of cars sold is approximately 96 cars per month during the 12 month period.

b.) During what month does the average rate of cars sold equal the rate of change of car sales?

The average rate of cars sold equals the instantaneous rate of chage of car sales when $t \approx 3.291$.

This would occur during the 4th month.