Skill Builder: Topic 5.2 – Extreme Value Theorem; Global vs Local Extrema, Critical Points

When completed properly, the table below will reveal a portion of a quote made famous by one of the founders of calculus, Gottfried Wilhelm Leibniz. To unveil the letters, answer each multiple choice question correctly and place the appropriate letter in the square that corresponds to the question number. Some problem numbers may appear more than once.

13	8	12		3	4	12	6	12	9	13		7	6				
T	H	E		P	R	E	S	E	N	T		I	S				
10	7	1		5	7	13	8		13	8	12						
B	I	G		W	I	T	H		T	H	E						
2	11	13	11	4	12												
F	U	T	U	R	E												

1. How many critical points does the function $f(x)=(x+2)^5(x-3)^4$ have?

(B) One (D) Two (G) Three (T) Five

$f'(x)=5(x+2)^4(x-3)^4+(x+2)^5\cdot 4(x-3)^3$

$(x+2)^4(x-3)^3\left[5(x-3)+4(x+2)\right]=0$

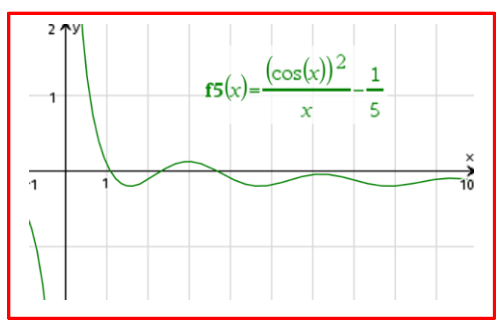
$(x+2)^4(x-3)^3(9x-7)=0$

There will be 3 critical points.

2. The first derivative of the function f is given by $f'(x)=\dfrac{\cos^2 x}{x}-\dfrac{1}{5}$. How many critical values does f have on the open interval (0,10)?

(G) One (F) Three (R) Four (S) Five

The graph of f ' crosses the x-axis 3 times and is never undefined.



3. Let f be the function defined by $f(x) = k\sqrt{x} - \ln x$ for $x > 0$, where k is a positive constant.

For what value of k does f have a critical point at $x = 1$?

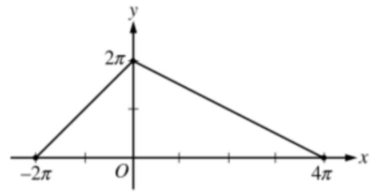
(D) $k = -1$ (G) $k = 1$ (L) $k = -2$ (P) $k = 2$

$$f'(x) = \frac{1}{2}kx^{-1/2} - \frac{1}{x}$$

If $f(x)$ has a critial point at $x = 1$, then $f'(1) = 0$

$$f'(1) = \frac{k}{2} \cdot \frac{1}{\sqrt{1}} - \frac{1}{1} = 0$$

$$\frac{k}{2} - 1 = 0 \to \frac{k}{2} = 1 \to k = 2$$



Graph of g

4. Let g be the piecewise-linear function defined on $[-2\pi, 4\pi]$ whose graph is given above, and let $f(x) = g(x) - \cos\left(\frac{x}{2}\right)$. Find all x-values in the open interval $(-2\pi, 4\pi)$ for which f has a critical point.

(R) $x = \pi$ and 0 (S) $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ (T) $x = \frac{\pi}{2}$ and π (W) $x = -\pi$ and π

$$f'(x) = g'(x) + \frac{1}{2} \cdot \sin\left(\frac{x}{2}\right)$$

$$f'(x) = 0 \text{ when } g'(x) = -\frac{1}{2} \cdot \sin\left(\frac{x}{2}\right)$$

This can only occur when either $1 = -\frac{1}{2} \cdot \sin\left(\frac{x}{2}\right)$ or $-\frac{1}{2} = -\frac{1}{2} \cdot \sin\left(\frac{x}{2}\right)$

$$-2 = \sin\left(\frac{x}{2}\right) \quad \text{or} \quad 1 = \sin\left(\frac{x}{2}\right)$$

No Solution or $\frac{x}{2} = \cdots, -\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2} \cdots$

The only solution on the interval $[-2\pi, 4\pi]$ is $x = \pi$

$f'(x)$ is undefined when $g'(x)$ is undefined which occurs when $g(x)$ is not differentiable. This occurs specifically at $x = 0$.

5. The function f is defined for all x in the closed interval $[a,b]$. If f does not attain a maximum value on $[a,b]$, which of the following must be true?

(R) f is continuous on $[a,b]$.

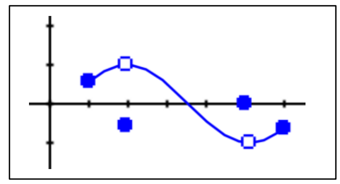
(S) f is not bounded above and below on $[a,b]$.

(T) f does not attain a minimum value on $[a,b]$.

(Y) The graph of f has a vertical asymptote in the interval $[a,b]$.

(W) None of the above

Choice (R) must be false, otherwise the EVT would apply and f would attain a maximum value.
Choice (T) could be false if the curve featured its lowest position at an undefined value of f for a specific x, yet define a value for f at that same x higher as shown in the topmost counterexample to the right.
Choices (S) and (Y) could be false using the same graphical counterexample to the right.



6. If f is a continuous function on the closed interval $[a,b]$, which of the following must be true?

(L) There is a number c in the open interval (a,b) such that $f(c)=0$.

(T) There is a number c in the open interval (a,b) such that $f(a)<f(c)<f(b)$.

(S) There is a number c in the open interval $[a,b]$ such that $f(c)\geq f(x)$ for all x in $[a,b]$.

(R) There is a number c in the open interval (a,b) such that $f'(c)=0$.

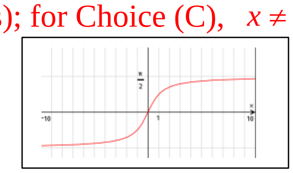
(E) There is a number c in the open interval (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Choice (A) could be true, but doesn't have to be because the graph of f does not have to cross the x-axis.
Choice (B) could be true, but doesn't have to be because the graph of f could be a horizontal line.
Choice (D) could be true, but doesn't have to be because Rolle's Theorem need not apply…i.e. $f(a)$ does not have to be equal to $f(b)$
Choice (E) could be true, but doesn't have to be because the Men Value Theorem need not apply because f may not be differentiable.
Choice (C) is true because the Extreme Value Theorem must apply.

7. Which of the following functions of x is guaranteed by the Extreme Value Theorem to have an absolute maximum on the interval [0,4]?

(A) $y=\tan x$ (I) $y=\tan^{-1} x$ (E) $y=\dfrac{x^2-16}{x^2+x-20}$ (O) $y=\dfrac{1}{e^x-1}$

In order for the Extreme Value Theorem to apply, the function must be continuous on the closed interval [0,4]. Choices (A), (C) and (D) each have values of x which would make their denominators equal to zero. (For choice (A), $x\neq\dfrac{\pi}{2}$ (among other values); for Choice (C), $x\neq 4$; for Choice (D), $x\neq 0$.



Choice (B) is the only function that is continuous.

8. Which of the following functions of x is guaranteed by the Extreme Value Theorem to have an absolute maximum on the interval $[0,2\pi]$?

(E) $y=\dfrac{1}{1+\sin x}$ (H) $y=\dfrac{1}{x^2+\pi}$ (T) $y=\dfrac{x^2-2\pi x+\pi^2}{x-\pi}$ (S) $y=\dfrac{|x-\pi|}{x-\pi}$

In order for the Extreme Value Theorem to apply, the function must be continuous on the closed interval $[0,2\pi]$. Choices (A), (C) and (D) each have values of x which would make their denominators equal to zero. (For choice (A), $x\neq\dfrac{3\pi}{2}$ (among other values); for Choice (C), $x\neq\pi$; for Choice (D), $x\neq\pi$.

Choice (B) is the only function that is continuous.

9. Let g be the function given by $g(x)=\sqrt{1-\cos x}$. Which of the following statements could be false on the interval $\dfrac{\pi}{2}\leq x\leq\dfrac{7\pi}{4}$?

(E) By the Extreme Value Theorem, there is a value c such that $g(c)\leq g(x)$ for $\dfrac{\pi}{2}\leq x\leq\dfrac{7\pi}{4}$.

(L) By the Extreme Value Theorem, there is a value c such that $g(c)\geq g(x)$ for $\dfrac{\pi}{2}\leq x\leq\dfrac{7\pi}{4}$.

(N) By the Intermediate Value Theorem, there is a value c such that $g(c)=\dfrac{g\left(\frac{\pi}{2}\right)+g\left(\frac{7\pi}{4}\right)}{2}$.

(T) By the Mean Value Theorem, there is a value c such that $g'(c)=\dfrac{g\left(\frac{7\pi}{4}\right)-g\left(\frac{\pi}{2}\right)}{\frac{7\pi}{4}-\frac{\pi}{2}}$.

Because $g(x)=\sqrt{1+\cos x}$ is both continuous on $\dfrac{\pi}{2}\leq x\leq\dfrac{7\pi}{4}$, the Extreme Value Theorem applies which means Choices (A) and (B) are always true.

Because $g(x)=\sqrt{1+\cos x}$ is both continuous on $\dfrac{\pi}{2}\leq x\leq\dfrac{7\pi}{4}$ and differentiable on $\dfrac{\pi}{2}<x<\dfrac{7\pi}{4}$, the Mean Value Theorem applies which mean Choice (D) is always true.

Choice (C) is not necessarily true, and besides, it does not correctly identify the conclusion of the Intermediate Value Theorem.

x	0	1	2	3
$f(x)$	0	4	7	6

10. Let f be a function with selected values given in the table above. Which of the following statements must be true?

I. By the Intermediate Value Theorem, there is a c in the interval (0,3) such that $f(c)=2$.
II. By the Mean Value Theorem, there is a c in the interval (0,3) such that $f'(c)=2$.
III. By the Extreme Value Theorem, there is a c in the interval [0,3] such that $f(c)\le f(x)$ for all x in the interval [0,3].

(B) None   (F) I only   (J) II only   (P) I, II, and III

All three of the statements do not have to be true based on the given information. The reason for this is due to the fact that all the theorems, IVT, MVT and EVT, require that the function be continuous. The problem did not say that f was continuous.

11. Let g be the function given by $g(x)=\sqrt{1-\sin^2 x}$. Which of the following statements could be false on the interval $\frac{\pi}{2}\le x\le\pi$?

(A) By the Extreme Value Theorem, there is a value c such that $g(c)\le g(x)$ for $\frac{\pi}{2}\le x\le\pi$.

(E) By the Extreme Value Theorem, there is a value c such that $g(c)\ge g(x)$ for $\frac{\pi}{2}\le x\le\pi$.

(I) By the Mean Value Theorem, there is a value c such that $g'(c)=\dfrac{g(\pi)-g\left(\frac{\pi}{2}\right)}{\pi-\frac{\pi}{2}}$.

(U) By the Intermediate Value Theorem, there is a value c such that $g(c)=\dfrac{g\left(\frac{\pi}{2}\right)+g(\pi)}{2}$.

Both Choices (A) and (B) must be true because $g(x)=\sqrt{1-\sin^2 x}$ is continuous on $\frac{\pi}{2}\le x\le\pi$

Both Choice (C) must be true because $g(x)=\sqrt{1-\sin^2 x}$ is continuous on $\frac{\pi}{2}\le x\le\pi$ and differentiable on $\frac{\pi}{2}<x<\pi$. Again, Choice (D) is not necessarily true, and besides, it does not correctly identify the conclusion of the Intermediate Value Theorem.

x	0	1	2	3
$f(x)$	15	14	12	9

12. Let f be a function with selected values given in the table above. Which of the following statements must be true?

I. By the Intermediate Value Theorem, there is a c in the interval (0,3) such that $f(c)=10$.
II. By the Mean Value Theorem, there is a c in the interval (0,3) such that $f'(c)=-2$.
III. By the Extreme Value Theorem, there is a c in the interval [0,3] such that $f(c)\le f(x)$ for all x in the interval [0,3].

(A) I, II, and III (O) I only (I) II only (E) None

All three of the statements do not have to be true based on the given information. The reason for this is due to the fact that all the theorems, IVT, MVT and EVT, require that the function be continuous. The problem did not say that f was continuous.

13. The function g is continuous on the closed interval [1,4] with $g(1)=5$ and $g(4)=8$. Of the following conditions, which could guarantee that there is a number c in the open interval (1,4) where $g'(c)=1$?

(L) g is increasing on the closed interval [1,4].

(T) g is differentiable on the open interval (1,4).

(S) g has a maximum value on the closed interval [1,4].

(W) The graph of g has at least one horizontal tangent in the open interval (1,4).

The Mean Value Theorem will only result in $g'(c)=\dfrac{g(4)-g(1)}{4-1}=\dfrac{8-5}{4-1}=1$ as long as g is continuous on the closed interval [1,4] and differentiable on the open interval (1,4).