Topic 5.5 – Using the Candidates Test to Determine Absolute (Global) Extrema

Find the absolute (global) maximum and absolute (global) minimum of the given function over the provided interval.

1.) $f(x) = 4x^2 - 4x + 1 \qquad [0, 2]$

$f'(x) = 8x - 4$

$f'(x) = 0$ | $f'(x)$ is undefined

$8x - 4 = 0$ | $\varnothing$

$x = \frac{1}{2}$

Candidates: $0, \frac{1}{2}$, and 2

x	$f(x)$
0	$4(0)^2 - 4(0) + 1 = 1$
$\frac{1}{2}$	$4\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 1 = 0$
2	$4(2)^2 - 4(2) + 1 = 9$

The absolute maximum is 9 which occurs when $x = 2$.

The absolute minimum is 0 which occurs when $x = \frac{1}{2}$.

2.) $f(x) = 6x^3 - 6x^4 + 5 \qquad [-1, 2]$

$f'(x) = 18x^2 - 24x^3$

$f'(x) = 0$ | $f'(x)$ is undefined

$18x^2 - 24x^3 = 0$ | $\varnothing$

$6x^2(3 - 4x) = 0$

$x = 0, \frac{3}{4}$

Candidates: $-1, 0, \frac{3}{4}$, and 2

x	$f(x)$
-1	$6(-1)^3 - 6(-1)^4 + 5 = -7$
0	$6(0)^3 - 6(0)^4 + 5 = 5$
$\frac{3}{4}$	$6\left(\frac{3}{4}\right)^3 - 6\left(\frac{3}{4}\right)^4 + 5 = \frac{721}{128}$
2	$6(2)^3 - 6(2)^4 + 5 = -43$

The absolute maximum is $\frac{721}{128}$ which occurs when $x = \frac{3}{4}$.

The absolute minimum is -43 which occurs when $x = 2$.

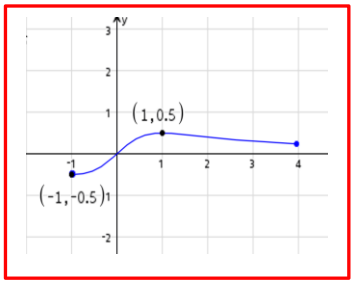
3.) $f(x)=\dfrac{x}{x^2+1}\quad[-1,4]$

$$f'(x)=\frac{1(x^2+1)-x(2x)}{(x^2+1)^2}=\frac{-x^2+1}{(x^2+1)^2}$$

$f'(x)=0$	$f'(x)$ is undefined
$-x^2+1=0$	$(x^2+1)^2=0$
$x=-1,\ 1$	$\varnothing$

Candidates: $-1, 1,$ and 4

x	$f(x)$
-1	$\dfrac{(-1)}{(-1)^2+1}=-\dfrac{1}{2}$
1	$\dfrac{1}{1^2+1}=\dfrac{1}{2}$
4	$\dfrac{4}{(4)^2+1}=-\dfrac{4}{17}$



The absolute maximum is $\frac{1}{2}$ which occurs when $x = 1$.

The absolute minimum is $-\frac{1}{2}$ which occurs when $x = -1$.

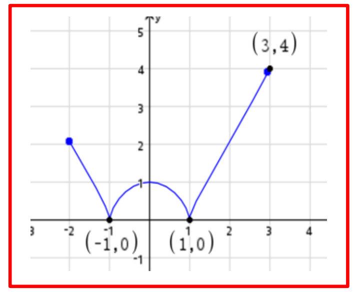
4.) $f(x)=\left(x^2-1\right)^{\frac{2}{3}}\quad[-2,3]$

$$f'(x)=\frac{2}{3}\left(x^2-1\right)^{-1/3}(2x)=\frac{4x}{3\sqrt[3]{x^2-1}}$$

$f'(x)=0$	$f'(x)$ is undefined
$4x=0$	$3\sqrt[3]{x^2-1}=0$
$x=0$	$x=-1,\ 1$

Candidates: $-1, 0,$ and 1

x	$f(x)$
-2	$\left((-2)^2-1\right)^{\frac{2}{3}}=\sqrt[3]{9}$
-1	$\left((-1)^2-1\right)^{\frac{2}{3}}=0$
0	$\left((0)^2-1\right)^{\frac{2}{3}}=1$
1	$\left((1)^2-1\right)^{\frac{2}{3}}=0$
3	$\left((3)^2-1\right)^{\frac{2}{3}}=4$



The absolute maximum is 4 which occurs when $x = 3$.
The absolute minimum is 0 which occurs when $x = -1$ and $x = 1$.

5.) $f(x)=x^{2/3}(20-x)\quad [-1,20]$

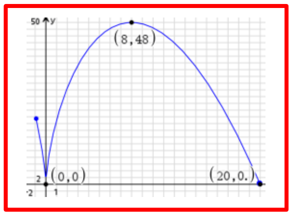
$f(x)=20x^{2/3}-x^{5/3}$

$f'(x)=\frac{40}{3}x^{-1/3}-\frac{5}{3}x^{2/3}=\frac{5}{3}x^{-1/3}(8-x)=\frac{5(8-x)}{3\sqrt[3]{x}}$

$f'(x)=0$	$f'(x)$ is undefined
$5(8-x)=0$	$3\sqrt[3]{x}=0$
$x=8$	$x=0$

Candidates: $-1, 0, 8,$ and 20

x	$f(x)$
-1	$(-1)^{2/3}(20-(-1))=21$
0	$(0)^{2/3}(20-(0))=0$
8	$(8)^{2/3}(20-(8))=48$
20	$(20)^{2/3}(20-(20))=0$

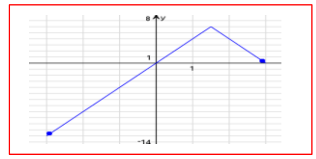


The absolute maximum is 48 which occur occurs when $x = 8$.
The absolute minimum is 0 which occurs when $x = 0$ and $x = 20$.

6.) $f(x)=6-|6-4x|\quad [-3,3]$

$$6-|6-4x|=\begin{cases}6-(6-4x), & 6-4x\ge 0\\ 6--(6-4x), & 6-4x<0\end{cases}=\begin{cases}4x, & x\le \frac{3}{2}\\ 12-4x, & x>\frac{3}{2}\end{cases}$$

$$f'(x)=\begin{cases}4, & x<\frac{3}{2}\\ -4, & x>\frac{3}{2}\end{cases}$$



$f'(x)=0$	$f'(x)$ is undefined
$\varnothing$	$x=\frac{3}{2}$

Candidates: $-3, \frac{3}{2},$ and 3

x	$f(x)$
-3	$6-\lvert 6-4(-3)\rvert=-12$
$\frac{3}{2}$	$6-\left\lvert 6-4\left(\frac{3}{2}\right)\right\rvert=6$
3	$6-\lvert 6-4(3)\rvert=0$

The absolute maximum is 6 which occur occurs when $x = 3/2$.

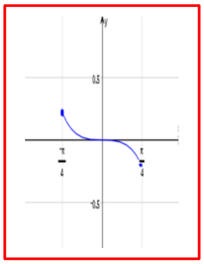
The absolute minimum is -12 which occurs when $x = -3$.

7.) $f(x) = x - \tan x \quad \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

$f'(x) = 1 - \sec^2 x = 1 - \frac{1}{\cos^2 x} = \frac{\cos^2 x - 1}{\cos^2 x}$

$f'(x) = 0$	$f'(x)$ is undefined
$\cos^2 x - 1 = 0$	$\cos^2 x = 0$
$\cos^2 x = 1$	$\varnothing$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
$\cos x = 1 \quad \cos x = -1$	
$x = 0$	Candidates: $-\frac{\pi}{4}, 0,$ and $\frac{\pi}{4}$

x	***f* (*x*)**
$-\frac{\pi}{4}$	$\left(-\frac{\pi}{4}\right) - \tan\left(-\frac{\pi}{4}\right) = -\frac{\pi}{4} + 1$
0	$(0) - \tan(0) = 0$
$\frac{\pi}{4}$	$\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - 1$



The absolute maximum is $-\frac{\pi}{4} + 1$ which occur occurs when $x = -\pi/4$.

The absolute minimum is $\frac{\pi}{4} - 1$ which occurs when $x = \pi/4$.

8.) $f(x) = \sin x - \cos x \quad [0, \pi]$

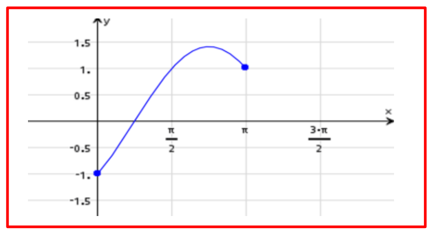
$f'(x) = \cos x + \sin x$

$f'(x) = 0$	$f'(x)$ is undefined
$\cos x + \sin x = 0$	$\varnothing$
$\cos x = -\sin x$	
$x = \frac{3\pi}{4}$	Candidates: $0, \frac{3\pi}{4},$ and π

x	***f* (*x*)**
0	$\sin(0) - \cos(0) = -1$
$\frac{3\pi}{4}$	$\sin\left(\frac{3\pi}{4}\right) - \cos\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$
π	$\sin(\pi) - \cos(\pi) = 1$

The absolute maximum is $\sqrt{2}$ which occur occurs when $x = 3\pi/4$.

The absolute minimum is -1 which occurs when $x = 0$.



9.) What is the smallest slope of the function

$y = x^3 - 3x^2 + 5x - 1$ on $\left[-\frac{1}{2}, 2\right]$?

We want to find the minimum of y'

$f'(x) = 3x^2 - 6x + 5$

$f''(x) = 6x - 6$

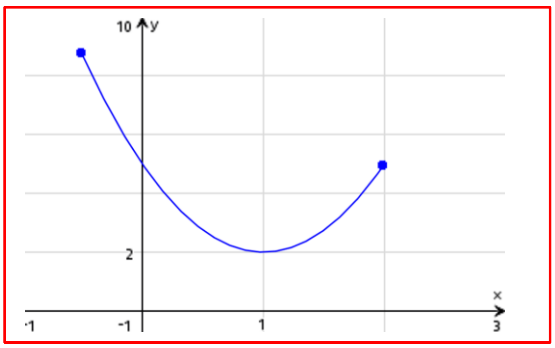
$\underline{f'(x) = 0}$ | $\underline{f'(x) \text{ is undefined}}$

$6x - 6 = 0$ | $\varnothing$

$x = 1$

Candidates: $-\frac{1}{2}, 1$, and 2

x	slope of $f(x)$ or $f'(x)$
$-\frac{1}{2}$	$3\left(-\frac{1}{2}\right)^2 - 6\left(-\frac{1}{2}\right) + 5 = \frac{3}{4} + 8 = \frac{35}{4}$
1	$3(1)^2 - 6(1) + 5 = 2$
2	$3(2)^2 - 6(2) + 5 = 5$



The minimum slope of $f(x)$ is 2 which occurs at $x = 1$.

10.) What is the largest slope of $y = \frac{x^2}{x^2+1}$ on $[0,10]$

We want to find the maximum of y'

$$f'(x) = \frac{2x(x^2+1) - x^2(2x)}{(x^2+1)^2} = \frac{2x^3 + 2x - 2x^3}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$$

$$f''(x) = \frac{2(x^2+1)^2 - 2x\cdot(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2(x^2+1)^2 - 8x^2(x^2+1)}{(x^2+1)^4}$$

$$= \frac{2(x^2+1)\left[(x^2+1) - 4x^2\right]}{(x^2+1)^4} = \frac{2(-3x^2+1)}{(x^2+1)^3} = \frac{-2(3x^2-1)}{(x^2+1)^3}$$

$\underline{f'(x) = 0}$ | $\underline{f'(x) \text{ is undefined}}$

$-2(3x^2 - 1) = 0$ | $(x^2+1)^3 = 0$

$x = -\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$ | $\varnothing$

Candidates: $0, \frac{1}{\sqrt{3}}$, and 10

x	slope of $f(x)$ or $f'(x)$
0	$\frac{2(0)^2}{\left((0)^2+1\right)^2} = 0$
$\frac{1}{\sqrt{3}}$	$\frac{2\left(\frac{1}{\sqrt{3}}\right)^2}{\left(\left(\frac{1}{\sqrt{3}}\right)^2+1\right)^2} = \frac{\frac{2}{3}}{\frac{16}{9}} = \frac{3}{8}$
10	$\frac{2(10)^2}{\left((10)^2+1\right)^2} = \frac{200}{10{,}201}$

The maximum slope of $f(x)$ is $\frac{3}{8}$ which occurs at $x = \frac{1}{\sqrt{3}}$.

In the following problems, the range of the periodic function $f(x)$ is the interval $[a,b]$ while the domain is $[0,2\pi]$. Find $b-a$.

11.) $f(x)=\sin x\cos x$

$f'(x)=\cos x\cdot\cos x+\sin x(-\sin x)$

$=\cos^2 x-\sin^2 x$

$\cos^2 x-\sin^2 x=\cos 2x$ by the Double Angle Identity

$\underline{f'(x)=0}$ $\quad$ $\underline{f'(x) \text{ is undefined}}$

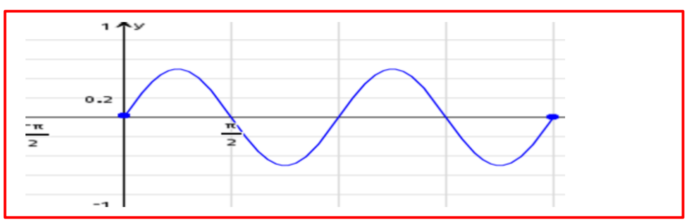
$\cos 2x=0$ $\quad$ $\varnothing$

$2x=\cdots,\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{7},\frac{7\pi}{2},\cdots$

$x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{5},\frac{7\pi}{4}$

Candidates: $0,\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{5},\frac{7\pi}{4}$ and 2π

x	$f(x)$
0	$\sin(0)\cdot\cos(0)=0$
$\frac{\pi}{4}$	$\sin\left(\frac{\pi}{4}\right)\cdot\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt2}\cdot\frac{1}{\sqrt2}=\frac{1}{2}$
[illegible]	$\sin\left(\frac{3\pi}{4}\right)\cdot\cos\left(\frac{3\pi}{4}\right)=\frac{1}{\sqrt2}\cdot-\frac{1}{\sqrt2}=-\frac{1}{2}$
$\frac{5\pi}{4}$	$\sin\left(\frac{5\pi}{4}\right)\cdot\cos\left(\frac{5\pi}{4}\right)=-\frac{1}{\sqrt2}\cdot-\frac{1}{\sqrt2}=\frac{1}{2}$
$\frac{7\pi}{4}$	$\sin\left(\frac{7\pi}{4}\right)\cdot\cos\left(\frac{7\pi}{4}\right)=-\frac{1}{\sqrt2}\cdot\frac{1}{\sqrt2}=-\frac{1}{2}$
2π	$\sin(2\pi)\cdot\cos(2\pi)=0$



The difference between the maximum and the minimum is $\frac{1}{2}-\left(-\frac{1}{2}\right)=1$.

12.) $f(x)=\cos x-\sin^2 x$

$f'(x)=-\sin x-2\sin x\cdot\cos x$

$\underline{f'(x)=0}$ $\quad$ $\underline{f'(x)\text{ is undefined}}$

$-\sin x-2\sin x\cdot\cos x=0$ $\quad$ $\varnothing$

$-\sin x(1+2\cos x)=0$

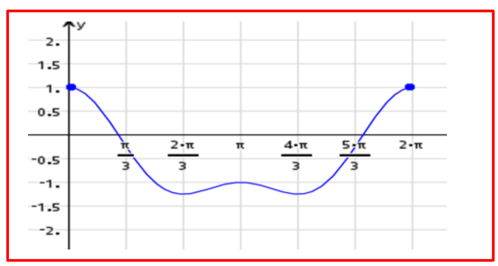
$-\sin x=0$ $\quad$ $1+2\cos x=0$

$\cos x=-\frac{1}{2}$

$x=0,\pi,2\pi$ $\quad$ $x=\frac{2\pi}{3},\frac{4\pi}{3}$

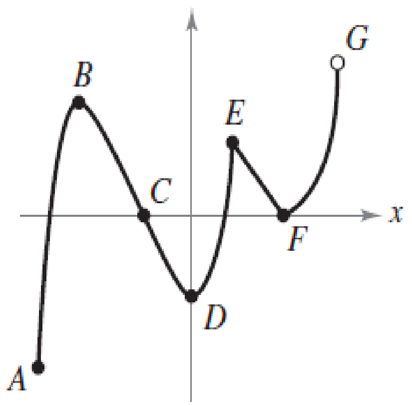
Candidates: $0,\frac{2\pi}{3},\pi,\frac{4\pi}{3}$, and 2π

x	$f(x)$
0	$\cos(0)-\sin^2(0)=1$
$\frac{2\pi}{3}$	$\cos\left(\frac{2\pi}{3}\right)-\sin^2\left(\frac{2\pi}{3}\right)=-\frac{1}{2}-\left(\frac{\sqrt3}{2}\right)^2=-\frac{5}{4}$
π	$\cos(\pi)-\sin^2(\pi)=-1$
$\frac{4\pi}{3}$	$\cos\left(\frac{4\pi}{3}\right)-\sin^2\left(\frac{4\pi}{3}\right)=-\frac{1}{2}-\left(\frac{\sqrt3}{2}\right)^2=-\frac{5}{4}$
2π	$\cos(2\pi)-\sin^2(2\pi)=1$



The difference between the maximum and the minimum is $1-\left(-\frac{5}{4}\right)=\frac{9}{4}$.

13.) Determine whether each labeled point is an absolute maximum or minimum, a relative maximum or minimum or neither.



A	**absolute minimum**
B	**relative maximum**
C	**point of inflection**
D	**relative minimum**
E	**relative maximum**
F	**relative minimum**
G	**neither**

14.) Determine whether each statement is True or False. If a statement is false, explain why or give an example that shows it to be false.

a.) The maximum of a function that is continuous on a closed interval can occur at two different values in the interval.

True. (Many of the exercises found on this Skill Builder have demonstrated this.)

b.) If a function is continuous on a closed interval, then it must have a minimum on the interval.

True. (The is the conclusion of the Extreme Value Theorem.)

c.) If $x = c$ is a critical number of the function f, then it is also a critical number of the function $g(x) = f(x) + k$, where k is a constant.

True. (The constant value *k* will disappear after the first derivative is taken.)

d.) If $x = c$ is a critical number of the function f, then it is also a critical number of the function $g(x) = f(x-k)$, where k is a constant.

False. A horizontal shift of *k* units to the right will cause the critical value to move to the right as well.

e.) Let the function f be differentiable on an interval I containing c. If f has a maximum value at $x = c$, then $-f$ has a minimum value at $x = c$.

True. (The negative sign will reflect the graph around the *x*-axis.)

f.) A quartic function has a derivative defined by the cubic function $f'(x) = ax^3 + bx^2 + cx + d$ where $a \neq 0$ will always have exactly three critical numbers.

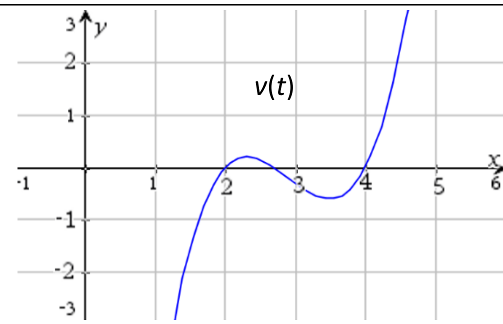
False. If the value(s) of *b* and/or *c* were equal to zero, there could be fewer unique solutions to $f'(x) = 0$. One such example would be $x^3 = 0$ which only has one unique solution, $x = 1$.

15.) A particle moves along the x-axis such that its position is $x(t)=0.25t^4-2.916t^3+12.25t^2-22t+21.6$ for $1\le t\le 4$. A graph of its velocity is shown to the right. At what time does the particle reach its leftmost position? Where is the particle when it reaches its leftmost position?

$$\frac{d}{dt}\left(0.25\cdot t^4-2.916\cdot t^3+12.25\cdot t^2-22\cdot t+21.6\right) \qquad t^3-8.748\cdot t^2+24.5\cdot t-22.$$

$$\text{solve}\left(t^3-8.748\cdot t^2+24.5\cdot t-22.=0,t\right) \qquad t=1.99475 \text{ or } t=2.76619 \text{ or } t=3.98707$$

$$0.25\cdot t^4-2.916\cdot t^3+12.25\cdot t^2-22\cdot t+21.6|t=\{1,1.99475,2.76619,3.98707,4\}$$
$$\{9.184,7.27198,7.39491,6.97579,6.976\}$$



The particle reaches its leftmost position at t = 3.987 at a location of approximately 6.975.

16.) If a particle moves along a straight line according to $s(t)=\frac{1}{12}t^4-\frac{1}{3}t^3-\frac{3}{2}t^2+4t-7$, find

a.) the maximum and minimum velocity on $0\le t\le 4$.

$v(t)=s'(t)=\frac{1}{3}t^3-t^2-3t+4$

$v'(t)=t^2-2t-3$

$v'(t)=0$ $v'(t)$ is undefined

$t^2-2t-3=0$ $\varnothing$

$(t-3)(t+1)=0$

$t=3,\ \not{-1}$

Candidates: 0, 3, and 4

t	$v(t)$
0	$\frac{1}{3}(0)^3-(0)^2-3(0)+4=4$
3	$\frac{1}{3}(3)^3-(3)^2-3(3)+4=-5$
4	$\frac{1}{3}(4)^3-(4)^2-3(4)+4=\frac{64}{3}-24=-\frac{8}{3}$

The minimum velocity is -5 which occurs when t = 3.
The maximum velocity is 4 which occurs when t = 0.

b.) the maximum and minimum acceleration on $0\le t\le 4$.

$a(t)=v'(t)=t^2-2t-3$

$a'(t)=2t-2$

$a'(t)=0$ $a'(t)$ is undefined

$2t-2=0$ $\varnothing$

$t=1$

Candidates: 0, 1, and 4

t	$v(t)$
0	$(0)^2-2(0)-3=-3$
1	$(1)^2-2(1)-3=-4$
4	$4^2-2(4)-3=5$

The minimum acceleration is -4 which occurs when t = 1.
The maximum acceleration is 5 which occurs when t = 4.