Topic 5.6 – Determining Concavity of Functions
Topic 5.7 – Using the Second Derivative Test

Determine the open intervals where the graph of the function is concave up or concave down. Identify any points of inflection. Use a number line to organize your analysis.

1.) $f(x) = x^4 - 6x^2 + 2x + 3$

$f'(x) = 4x^3 - 12x + 2$

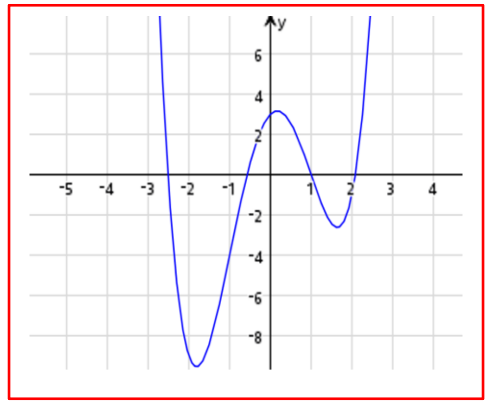
$f''(x) = 12x^2 - 12$

$\underline{f''(x) = 0}$ $\qquad$ $\underline{f''(x) \text{ is not differentiable}}$

$12x^2 - 12 = 0$ $\qquad$ $\varnothing$

$12(x^2 - 1) = 0$

$x = -1,\ 1$



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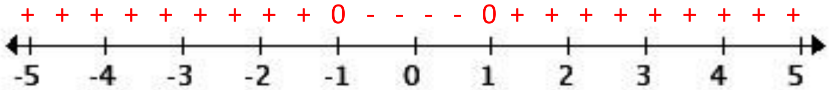


$f(x)$ is concave upward on $(-\infty, -1)$ and $(1, \infty)$ because $f''(x) > 0$ on those intervals.

$f(x)$ is concave downward on $(-1, 1)$ because $f''(x) < 0$ on that interval.

$f(x)$ has points of inflection at on $(-1, -4)$ and $(1, 0)$ because $f''(x)$ changes signs at $x = -1$ and $x = 1$.

2.) $f(x) = x + \dfrac{1}{x}$

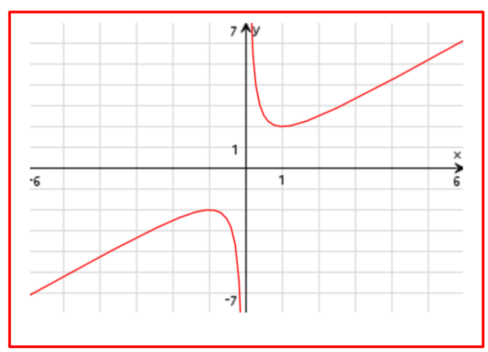
$f'(x) = 1 - x^{-2}$

$f''(x) = 2x^{-3} = \dfrac{2}{x^3}$

$\underline{f''(x) = 0}$ $\qquad$ $\underline{f''(x) \text{ is not differentiable}}$

$\varnothing$ $\qquad$ $x^3 = 0$

$x = 0$



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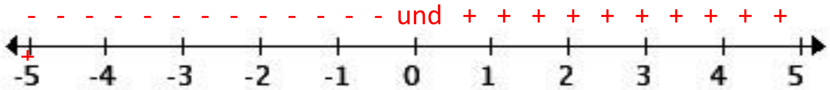
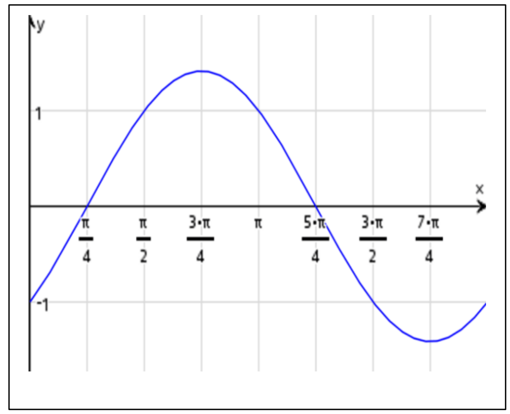


$f(x)$ is concave downward on $(-\infty, 0)$ because $f''(x) < 0$ on that interval.

$f(x)$ is concave upward on $(0, \infty)$ because $f''(x) > 0$ on that interval.

$f(x)$ has no points of inflection. Note: $f(0)$ is undefined

3.) $f(x) = \sin x - \cos x$ on $(0, 2\pi)$.



$f'(x) = \cos x + \sin x$

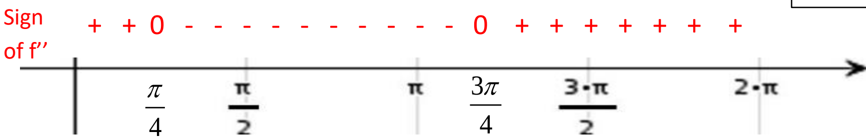
$f''(x) = -\sin x + \cos x$

$\underline{f''(x) = 0}$ $\qquad\qquad$ $\underline{f''(x) \text{ is not differentiable}}$

$-\sin x + \cos x = 0$ $\qquad\qquad$ $\varnothing$

$\cos x = \sin x$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

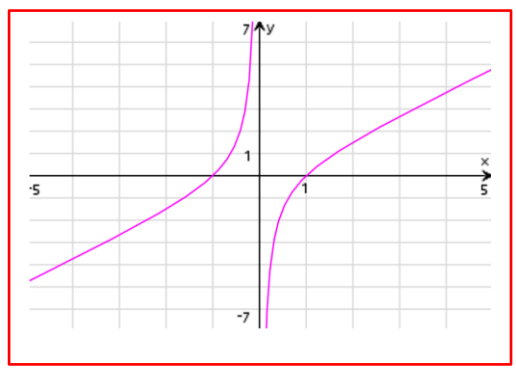


$f(x)$ is concave upward on $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right)$ because $f''(x) > 0$ on those intervals.

$f(x)$ is concave downward on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ because $f''(x) < 0$ on that interval.

$f(x)$ has points of inflection at on $\left(\frac{\pi}{4}, 0\right)$ and $\left(\frac{5\pi}{4}, 0\right)$ because $f''(x)$ changes signs at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

4.) $f(x) = \frac{x^2 - 1}{x} = x - x^{-1}$



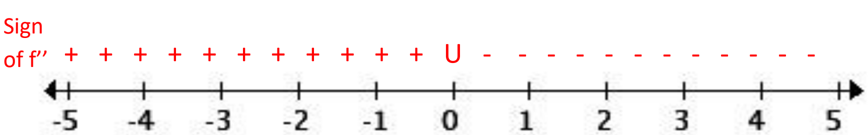
$f'(x) = 1 + x^{-2}$

$f''(x) = -2x^{-3} = -\frac{2}{x^3}$

$\underline{f''(x) = 0}$ $\qquad\qquad$ $\underline{f''(x) \text{ is not differentiable}}$

$\varnothing$ $\qquad\qquad$ $x^3 = 0$

$x = 0$

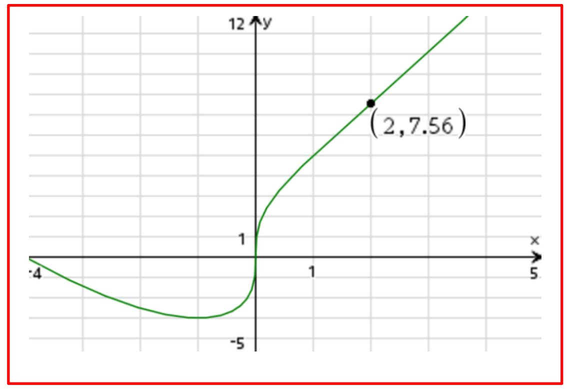


$f(x)$ is concave upward on $(-\infty, 0)$ because $f''(x) > 0$ on that interval.

$f(x)$ is concave downward on $(0, \infty)$ because $f''(x) < 0$ on that interval.

$f(x)$ has no points of inflection. Note: $f(0)$ is undefined

5.) $f(x)=x^{4/3}+4x^{1/3}$



$f'(x)=\frac{4}{3}x^{1/3}+\frac{4}{3}x^{-2/3}$

$f''(x)=\frac{4}{9}x^{-2/3}-\frac{8}{9}x^{-5/3}=\frac{4}{9}x^{-5/3}(x-2)=\frac{4(x-2)}{9\sqrt[3]{x^5}}$

$\underline{f''(x)=0}$ | $\underline{f''(x)\text{ is not differentiable}}$

$4(x-2)=0$ | $9\sqrt[3]{x^5}=0$

$x=2$ | $x=0$

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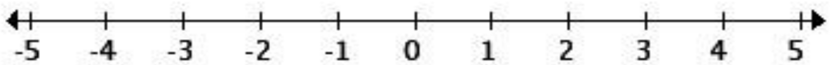
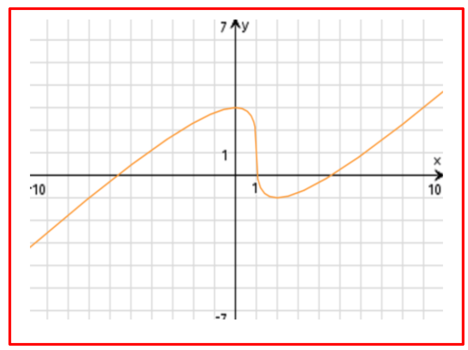


$f(x)$ is concave upward on $(-\infty,0)$ and $(2,\infty)$ because $f''(x)>0$ on those intervals.

$f(x)$ is concave downward on $(0,2)$ because $f''(x)<0$ on that interval.

point of inflection at $x=0$ and $x=2$ because $f''(x)$ changes signs at $x=0$ and $x=2$.

6.) $f(x)=x+3(1-x)^{1/3}$



$f'(x)=1+(1-x)^{-2/3}(-1)=1-(1-x)^{-2/3}$

$f''(x)=\frac{2}{3}(1-x)^{-5/3}(-1)=\frac{-2}{3(1-x)^{5/3}}$

$\underline{f''(x)=0}$ | $\underline{f''(x)\text{ is not differentiable}}$

$\varnothing$ | $3(1-x)^{5/3}=0$

$x=1$

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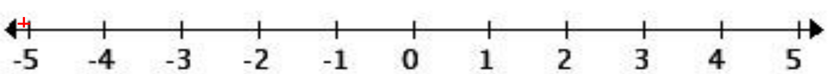
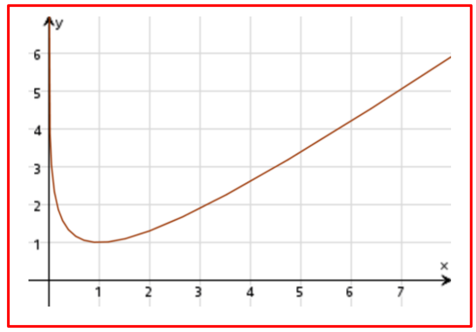


$f(x)$ is concave downward on $(-\infty,1)$ because $f''(x)<0$ on those intervals.

$f(x)$ is concave upward on $(1,\infty)$ because $f''(x)>0$ on that interval.

$f(x)$ has a point of inflection at $x=1$ because $f''(x)$ changes signs at $x=1$

7.) $f(x) = x - \ln x$ Note that $f(x)$ is only defined for $x > 0$



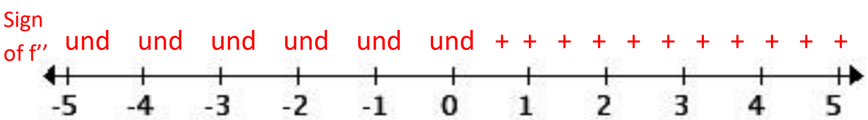
$f'(x) = 1 - \frac{1}{x}$

$f''(x) = \frac{1}{x^2}$

$\underline{f''(x) = 0}$ $\underline{f''(x) \text{ is not differentiable}}$

$\varnothing$ $x^2 = 0$

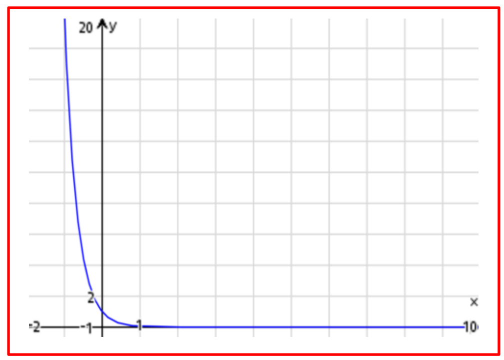
$x = 0$



$f(x)$ is concave upward on $(0, \infty)$ because $f''(x) > 0$ on that interval.

$f(x)$ has no points of inflection.

8.) $f(x) = e^{-3x}$



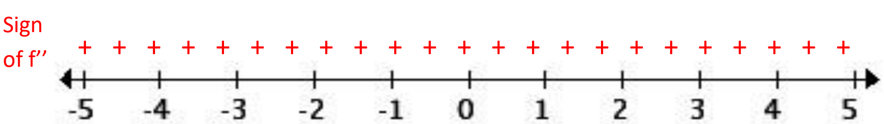
$f'(x) = -3e^{-3x}$

$f''(x) = 9e^{-3x} = \frac{9}{e^{3x}}$

$\underline{f''(x) = 0}$ $\underline{f''(x) \text{ is not differentiable}}$

$\varnothing$ $e^{3x} = 0$

$\varnothing$



$f(x)$ is concave upward on $(-\infty, \infty)$ because $f''(x) > 0$ everywhere

$f(x)$ has no points of inflection.

Find all critical numbers and use the Second Derivative Test to classify each as the location of a relative maximum, relative minimum or neither.

9.) $f(x) = x^4 + 4x^2 + 1$

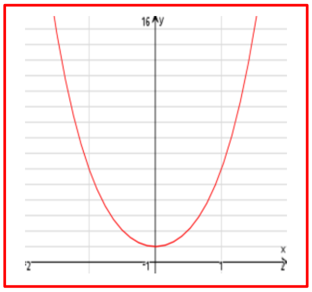
$f'(x) = 4x^3 + 8x$

$f'(x) = 0$

$4x(x^2 + 2) = 0$

$x = 0$

$f''(x) = 12x^2 + 8$



x	*f''(x)*
0	$12(0)^2 + 8 = 8$

$f(x)$ has a relative minimum at $x = 0$ because $f'(0) = 0$ and $f''(0) = 8 < 0$.

10.) $f(x) = \dfrac{x^2 - 1}{x}$

11.) $f(x) = x^{1/5}(x+1)$

12.) $f(x) = \dfrac{\sqrt{x}}{1+\sqrt{x}}$

13.) $f(x) = x^2 e^{-x}$

14.) $f(x) = \dfrac{x}{\ln x}$