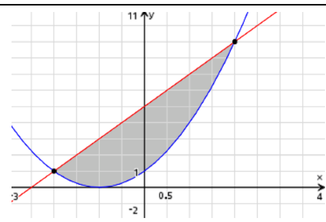
Skill Builder: Topics 8.4–8.6 – Area Between Curves

For each problem, sketch the region bounded by the graphs of the functions and find the area of the region. Problems marked a calculator icon indicate that you can use your calculator to evaluate the definite integral.

1.) $y = x^2 + 2x + 1, \quad y = 2x + 5$



$$A = \int_{-2}^{2} \left(2x+5-\left(x^2+2x+1\right)\right)dx$$

$$A = \int_{-2}^{2} \left(-x^2+4\right)dx$$

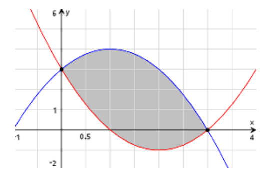
$$A = \left(-\frac{x^3}{3}+4x\right)\Bigg|_{-2}^{2}$$

$$A = -\frac{(2)^3}{3}+4(2)-\left(-\frac{(-2)^3}{3}+4(-2)\right)$$

$$A = -\frac{8}{3}+8-\frac{8}{3}+8 = 16-\frac{16}{3} = \frac{32}{3}$$

$$x^2+2x+1 = 2x+5$$
$$x^2-4 = 0$$
$$x = \pm 2$$

2.) $y = x^2 - 4x + 3, \quad y = -x^2 + 2x + 3$



$$A = \int_{0}^{3} \left(-x^2+2x+3-\left(x^2-4x+3\right)\right)dx$$

$$A = \int_{0}^{3} \left(-2x^2+6x\right)dx$$

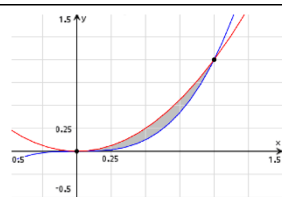
$$A = \left(-\frac{2x^3}{3}+3x^2\right)\Bigg|_{0}^{3}$$

$$A = -\frac{2(3)^3}{3}+3(3)^2-(0)$$

$$A = -\frac{54}{3}+27 = -18+27 = 9$$

$$x^2-4x+3 = -x^2+2x+3$$
$$2x^2-6x = 0$$
$$2x(x-3) = 0$$
$$x = 0, 3$$

3.) $y = x^2, \quad y = x^3$



$$A = \int_{0}^{1} \left(x^2-x^3\right)dx$$

$$A = \left(\frac{x^3}{3}-\frac{x^4}{4}\right)\Bigg|_{0}^{1}$$

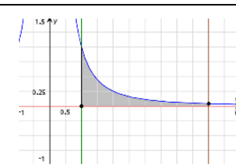
$$A = \frac{(1)^3}{3}-\frac{(1)^4}{4}-(0)$$

$$A = \frac{1}{3}-\frac{1}{4} = \frac{1}{12}$$

$$x^3 = x^2$$
$$x^3-x^2 = 0$$
$$x^2(x-1) = 0$$
$$x = 0, 1$$

<u>Note:</u> It may be difficult to determine which function lies above the other on the interval [0,1]. It is important to realize that a decimal **less than 1** is raised to the 3rd power, it results in a smaller value than it that number were raised to the second power.

4.) $y = \frac{1}{x^2}, \quad y = 0, \quad x = 1, \quad x = 5$



$$A = \int_{1}^{5} \frac{1}{x^2}\,dx$$

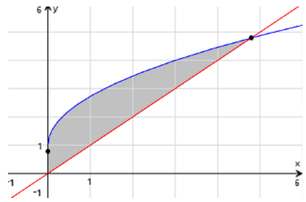
$$A = \left(-\frac{1}{x}\right)\Bigg|_{1}^{5}$$

$$A = -\frac{1}{5}-(-1) = \frac{4}{5}$$

5.) $y=\sqrt{3x}+1,\ y=x,\ x=0$

$$A=\int_0^{4.791}\left(\sqrt{3x}+1-x\right)dx$$

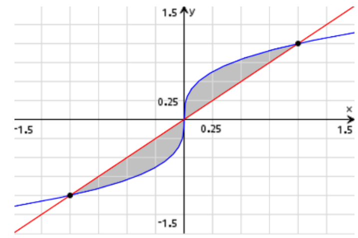
$$A=5.423$$



solve$\left(\sqrt{3\cdot x}+1=x,x\right)$ $\quad x=\dfrac{\sqrt{21}+5}{2}$

solve$\left(\sqrt{3\cdot x}+1=x,x\right)$ $\quad x=4.79129$

6.) $y=\sqrt[3]{x},\ y=x$



$\sqrt[3]{x}=x$

$x=x^3$

$x^3-x=0$

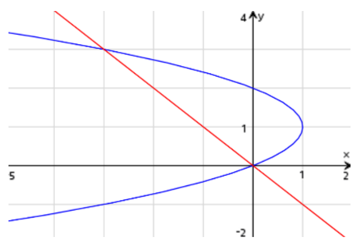
$x(x^2-1)=0$

$x=-1,0,1$

$$A=\int_{-1}^{0}\left(x-\sqrt[3]{x}\right)dx+\int_0^1\left(\sqrt[3]{x}-x\right)dx \text{ OR } 2\int_0^1\left(\sqrt[3]{x}-x\right)dx$$

$$A=2\left(\frac{3}{4}x^{4/3}-\frac{1}{2}x^2\right)\Bigg|_0^1=2\left(\frac{3}{4}-\frac{1}{2}\right)=\frac{1}{2}$$

7.) $x=2y-y^2,\ x=-y$



$2y-y^2=-y$

$y^2-3y=0$

$y(y-3)=0$

$y=0,3$

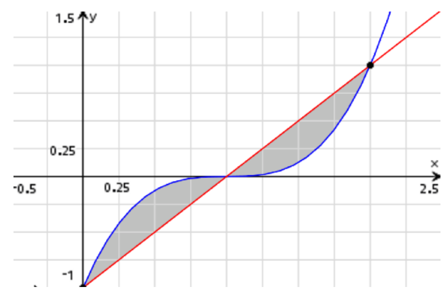
$$A=\int_0^3\left(2y-y^2-(-y)\right)dy$$

$$A=\int_0^3\left(-y^2+3y\right)dy$$

$$A=\left(-\frac{y^3}{3}+\frac{3y^2}{2}\right)\Bigg|_0^3$$

$$A=-\frac{27}{3}+\frac{27}{2}=\frac{-54+81}{6}=\frac{27}{6}=\frac{9}{2}$$

8.) $y=(x-1)^3,\ y=x-1$



$$A=\int_0^1\left((x-1)^3-(x-1)\right)dx+\int_1^2\left((x-1)-(x-1)^3\right)dx$$

$$A=\int_0^1\left(x^3-3x^2+3x-1-x+1\right)dx+\int_1^2\left(x-1-x^3+3x^2-3x+1\right)dx$$

$$A=\int_0^1\left(x^3-3x^2+2x\right)dx+\int_1^2\left(-x^3+3x^2-2x\right)dx$$

$$A=\left(\frac{x^4}{4}-x^3+x^2\right)\Bigg|_0^1+\left(-\frac{x^4}{4}+x^3-x^2\right)\Bigg|_1^2$$

$$A=\frac{1}{4}-1+1-\frac{16}{4}+8-4-\left(-\frac{1}{4}+1-1\right)$$

$$A=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

Note: A more convenient approach to solving this problem would be to recognize the symmetry that exists between the two shaded regions. Therefore, a single integral expression like

$$2\int_0^1\left((x-1)^3-(x-1)\right)dx$$

would have computed the correct area.

9.) $y = 2\sin x,\ y = \tan x$	**10.)** $y = \dfrac{1}{1+x^2},\ y = \dfrac{1}{2}x^2$

9.) $y = 2\sin x,\ y = \tan x$

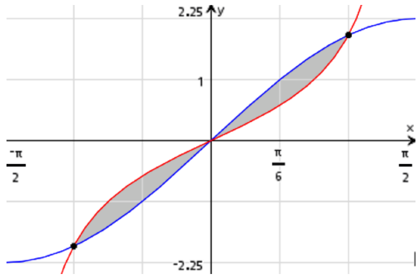
$2\sin x = \tan x$

$2\sin x = \dfrac{\sin x}{\cos x}$

$2 = \dfrac{1}{\cos x}$

$\cos x = \dfrac{1}{2}$

$x = -\dfrac{\pi}{3}, \dfrac{\pi}{3}$



$$A = \int_{-\frac{\pi}{3}}^{0} \left((\tan x - 2\sin x\right)dx + \int_{0}^{\frac{\pi}{3}} \left(2\sin x - \tan x\right)dx$$

Using your calculator,

$A = -2\left(\ln 2 - 1\right) \approx 0.614$

Note: A more convenient approach to solving this problem would be to recognize the symmetry that exists between the two shaded regions. Therefore, a single integral expression like

$$2\int_{0}^{\frac{\pi}{3}} \left(2\sin x - \tan x\right)dx$$

would have computed the correct area.

10.) $y = \dfrac{1}{1+x^2},\ y = \dfrac{1}{2}x^2$

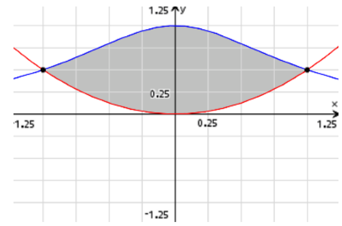
$\dfrac{1}{1+x^2} = \dfrac{x^2}{2}$

$x^4 + x^2 = 2$

$x^4 + x^2 - 2 = 0$

$(x^2+2)(x^2-1) = 0$

$x = -1, 1$



$$A = \int_{-1}^{1} \left(\frac{1}{1+x^2} - \frac{1}{2}x^2\right)dx$$

$$A = \left(\arctan x - \frac{1}{6}x^3\right)\Bigg|_{-1}^{1}$$

$$A = \arctan(1) - \frac{1}{6} - \left(\arctan(-1) - \left(-\frac{1}{6}\right)\right)$$

$$A = \frac{\pi}{4} - \frac{1}{6} - \left(-\frac{\pi}{4} + \frac{1}{6}\right)$$

$$A = \frac{\pi}{2} - \frac{1}{3}$$

Note: A more convenient approach to solving this problem would be to recognize the symmetry that exists between the two shaded regions. Therefore, a single integral expression like

$$2\int_{0}^{1} \left(\frac{1}{1+x^2} - \frac{1}{2}x^2\right)dx$$

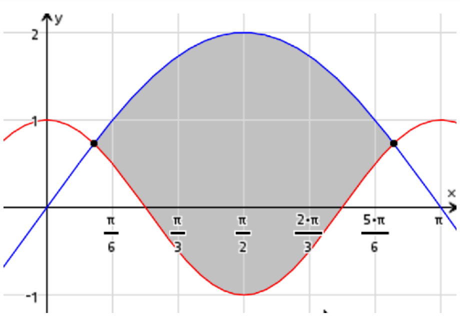
would have computed the correct area.

11.) $y = 2\sin x,\ y = \cos 2x$

Use your TI-Nspire to find intersection points as well.

solve$(2\cdot \sin(x) = \cos(2\cdot x), x)|0 < x < \pi$

$x = 0.374734$ or $x = 2.76686$



$$A = \int_{0.374734}^{2.76686} \left(2\sin x - \cos(2x)\right)dx$$

$A \approx 4.404$

12.) $y = x^3$ and the tangent to y at (1, 1)

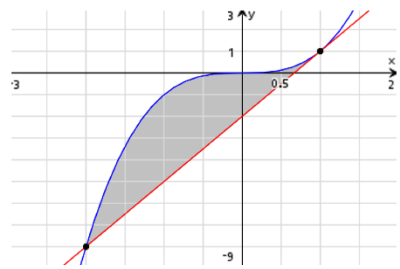
$y' = 3x^2$

$y'(1) = 3(1)^2 = 3$

$y(1) = 1^3 = 1$

$y - 1 = 3(x-1)$

$y = 3x - 2$



Find points of intersection.

$x^3 = 3x - 2$

$x^3 - 3x + 2 = 0$

This is a somewhat challenging equation to solve for *x*.
You can use synthetic division or trial and error to discover that x = -2 and 1.

$$A = \int_{-2}^{1} \left(x^3 - (3x-2\right)dx$$

$$A = \left(\frac{1}{4}x^4 - \frac{3}{2}x^2 + 2x\right)\Bigg|_{-2}^{1}$$

$$A = \frac{1}{4} - \frac{3}{2} + 2 - \left(4 - 6 - 4\right)$$

$$A = -\frac{5}{4} + 8 \text{ or } \frac{27}{4}$$

13.) Find the value(s) of b if the vertical line $x = b$ divides the region between $y = 16 - 2x$ and the x-axis and y-axis into 2 equal regions.

$$A = \int_0^8 (16 - 2x)\,dx$$

$$A = \left(16x - x^2\right)\Big|_0^8$$

$$A = 128 - 64$$

$$A = 64$$

$$A_{half} = \int_0^b (16 - 2x)\,dx = 32$$

$$A_{half} = \left(16x - x^2\right)\Big|_0^b = 32$$

$$A_{half} = 16b - b^2 = 32$$

$$b^2 - 16b + 32 = 0$$

$$b = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(1)(32)}}{2(1)} = \frac{16 \pm \sqrt{256 - 128}}{2}$$

$$b = \frac{16 \pm \sqrt{128}}{2}$$

Disregard the "+" as it results in a b value greater than 10 which lies outside the shaded region.

$$\therefore b = \frac{16 - \sqrt{128}}{2} \approx 2.343$$