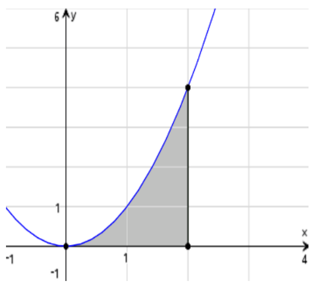
Skill Builder: Topics 8.7-8.8 – Volumes with Cross Sections

For each problem, sketch the region bounded by the graphs of the functions and find the area of the region. <u>Be sure to draw in the representative rectangle</u>. Only use a calculator for problems marked with the calculator icon.

1.) Find the volume of the solid whose base is the region bounded between the curve $y = x^2$ and the x-axis from $x = 0$ to $x = 2$ and whose cross sections taken perpendicular to the x-axis are depicted by each of the following shapes.



$s = x^2 - 0$

a.) squares

$$V = \int_0^2 (x^2)^2\, dx$$

$$= \left.\frac{x^5}{5}\right|_0^2$$

$$= \frac{32}{5}$$

b.) semicircles $r = \dfrac{x^2 - 0}{2}$

$$V = \frac{\pi}{2}\int_0^2 \left(\frac{x^2}{2}\right)^2 dx$$

$$= \left.\frac{\pi}{8}\cdot\frac{x^5}{5}\right|_0^2$$

$$= \frac{\pi}{8}\cdot\frac{32}{5} = \frac{4}{5}\pi$$

c.) rectangles whose heights are 3 times the base width

$$V = \int_0^2 (3x^2)(x^2)\, dx$$

$$= \int_0^2 3x^4 dx$$

$$= \left.\frac{3x^5}{5}\right|_0^2$$

$$= \frac{96}{5}$$

d.) equilateral triangles

$$V = \int_0^2 \frac{1}{2}(x^2)\left(\frac{\sqrt{3}}{2}x^2\right) dx$$

$$= \int_0^2 \frac{\sqrt{3}}{4}x^4\, dx$$

$$= \left.\frac{\sqrt{3}x^5}{4\cdot 5}\right|_0^2 = \frac{\sqrt{3}}{20}(32) = \frac{8\sqrt{3}}{5}$$

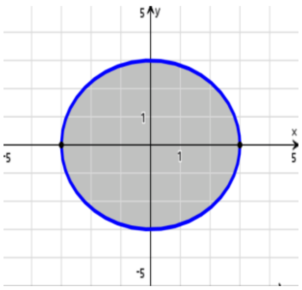
e.) isosceles right triangles where the base is the hypotenuse

$$V = \int_0^2 \frac{1}{2}(x^2)\left(\frac{1}{2}x^2\right) dx$$

$$= \int_0^2 \frac{1}{4}x^4\, dx$$

$$= \left.\frac{x^5}{4\cdot 5}\right|_0^2 = \frac{32}{20} = \frac{8}{5}$$

2.) Find the volume of the solid whose base is bounded by the circle whose center is the origin, whose radius is 3 and whose cross sections taken perpendicular to the *x*-axis are depicted by each of the following shapes.



Equation of circle

$$x^2 + y^2 = 9$$
$$y^2 = 9 - x^2$$
$$y = \pm\sqrt{9 - x^2}$$

$$s = \sqrt{9 - x^2} - \left(-\sqrt{9 - x^2}\right)$$
$$= 2\sqrt{9 - x^2}$$

a.) squares

$$V = \int_{-3}^{3} \left(2\sqrt{9 - x^2}\right)^2 dx \text{ or } 2\cdot\int_{0}^{3} \left(2\sqrt{9 - x^2}\right)^2 dx$$
$$= 8\int_0^3 (9 - x^2)\,dx$$
$$= 8\left(9x - \frac{x^3}{3}\right)\Bigg|_0^3 = 8(27 - 9) = 144$$

b.) semicircles $r = \dfrac{2\sqrt{9 - x^2}}{2} = \sqrt{9 - x^2}$

$$V = \frac{\pi}{2}\int_{-3}^{3} \left(\sqrt{9 - x^2}\right)^2 dx = \pi\int_0^3 \left(\sqrt{9 - x^2}\right)^2 dx$$
$$= \pi\int_0^3 (9 - x^2)\,dx$$
$$= \pi\left(9x - \frac{x^3}{3}\right)\Bigg|_0^3 = \pi(27 - 9) = 18\pi$$

c.) rectangles whose heights are twice the base width

$$V = \int_{-3}^{3} 2\left(2\sqrt{9 - x^2}\right)\left(2\sqrt{9 - x^2}\right)dx \text{ or } 2\int_0^3 2\left(2\sqrt{9 - x^2}\right)\left(2\sqrt{9 - x^2}\right)dx$$
$$= 16\int_0^3 \left(9 - x^2\right)dx$$
$$= 16\left(9x - \frac{x^3}{3}\right)\Bigg|_0^3 = 16(27 - 9)$$
$$= 288$$

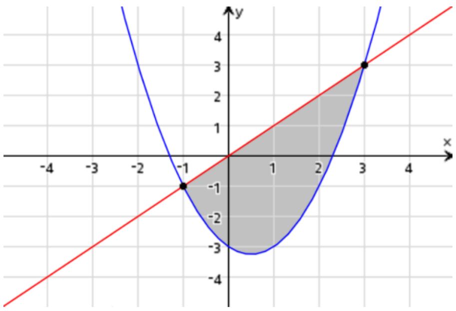
d.) equilateral triangles

$$V = \int_{-3}^{3} \frac{1}{2}\left(2\sqrt{9 - x^2}\right)\left(\frac{\sqrt{3}}{2}\cdot 2\sqrt{9 - x^2}\right)dx$$
$$\text{or } 2\int_0^3 \left(\sqrt{9 - x^2}\right)\left(\sqrt{3}\cdot\sqrt{9 - x^2}\right)dx$$
$$= 2\sqrt{3}\int_0^3 \left(9 - x^2\right)dx$$
$$= 2\sqrt{3}\left(9x - \frac{x^3}{3}\right)\Bigg|_0^3 = 2\sqrt{3}(27 - 9)$$
$$= 36\sqrt{3}$$

e.) isosceles right triangles where the base is the hypotenuse

$$V = \int_{-3}^{3} \frac{1}{2}\left(2\sqrt{9 - x^2}\right)\left(\frac{1}{2}\cdot 2\sqrt{9 - x^2}\right)dx$$
$$\text{or } 2\int_0^3 \frac{1}{2}\left(2\sqrt{9 - x^2}\right)\left(\frac{1}{2}\cdot 2\sqrt{9 - x^2}\right)dx$$
$$= 2\int_0^3 \left(9 - x^2\right)dx$$
$$= 2\left(9x - \frac{x^3}{3}\right)\Bigg|_0^3 = 2(27 - 9)$$
$$= 36$$

3.) Find the volume of the solid whose base is the region bounded between the curve $y = x^2 - x - 3$ and $y = x$ and whose cross sections taken perpendicular to the x-axis are depicted by each of the following shapes.



$s = x - (x^2 - x - 3) = -x^2 + 2x + 3$

a.) squares

$$V = \int_{-1}^{3} \left(-x^2 + 2x + 3\right)^2 dx$$

$$= \frac{512}{15}$$

$$\int_{-1}^{3} \left(-x^2 + 2 \cdot x + 3\right)^2 dx \qquad \frac{512}{15}$$

b.) semicircles $r = \dfrac{-x^2 + 2x + 3}{2}$

$$V = \frac{\pi}{2} \int_{-1}^{3} \left(\frac{-x^2 + 2x + 3}{2}\right)^2 dx =$$

$$= \frac{64\pi}{15}$$

$$\frac{\pi}{2} \cdot \int_{-1}^{3} \left(\frac{-x^2 + 2 \cdot x + 3}{2}\right)^2 dx \qquad \frac{64 \cdot \pi}{15}$$

c.) rectangles whose heights are half the base width

$$V = \int_{-1}^{3} \frac{1}{2}\left(-x^2 + 2x + 3\right)\left(-x^2 + 2x + 3\right) dx$$

$$= \frac{256}{15}$$

$$\int_{-1}^{3} \left(\frac{1}{2} \cdot \left(-x^2 + 2 \cdot x + 3\right)^2\right) dx \qquad \frac{256}{15}$$

d.) equilateral triangles

$$V = \frac{\sqrt{3}}{4} \int_{-1}^{3} \left(-x^2 + 2x + 3\right)^2 dx$$

$$= \frac{512\sqrt{3}}{60} = \frac{128\sqrt{3}}{15}$$

$$\frac{\sqrt{3}}{4} \cdot \int_{-1}^{3} \left(-x^2 + 2 \cdot x + 3\right)^2 dx \qquad \frac{128 \cdot \sqrt{3}}{15}$$

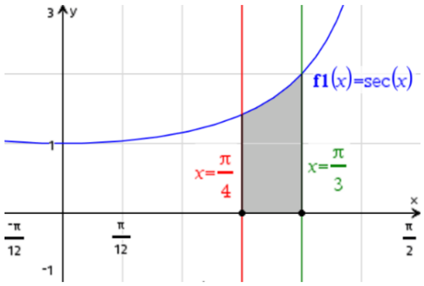
e.) isosceles right triangles where the base is the hypotenuse

$$V = \frac{1}{4} \int_{-1}^{3} \left(-x^2 + 2x + 3\right)^2 dx$$

$$= \frac{512}{60} = \frac{128}{15}$$

$$\frac{1}{4} \cdot \int_{-1}^{3} \left(-x^2 + 2 \cdot x + 3\right)^2 dx \qquad \frac{128}{15}$$

4.) Find the volume of the solid whose base is the region bounded between the curve $y = \sec x$ and the x-axis from $x = \pi/4$ to $x = \pi/3$ and whose cross sections taken perpendicular to the x-axis are squares.



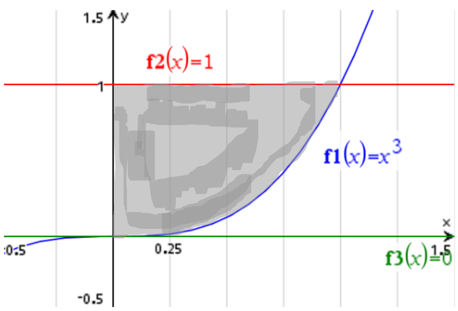
$$V = \int_{\pi/4}^{\pi/3} \left(\sec^2 x\right) dx$$

$$= \tan x \Big|_{\pi/4}^{\pi/3}$$

$$= \tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{4}\right)$$

$$= \sqrt{3} - 1$$

5.) Find the volume of the solid whose base is the region bounded between the curve $y = x^3$ and the y-axis from $y = 0$ to $y = 1$ and whose cross sections taken perpendicular to the y-axis are squares.



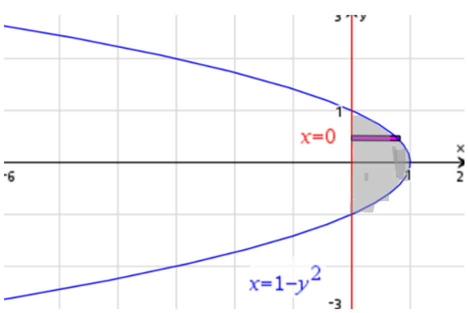
This will require a *dy* set-up.

$$y = x^3 \rightarrow x = \sqrt[3]{y}$$

$$V = \int_0^1 \left(\sqrt[3]{y}\right)^2 dy = \int_0^1 y^{2/3}\, dy$$

$$= \frac{y^{5/3}}{\frac{5}{3}}\Bigg|_0^1 = \frac{3}{5}(1-0) = \frac{3}{5}$$

6.) Find the volume of the solid whose base is the region bounded between the curve $x = 1 - y^2$ and the y-axis and whose cross sections taken perpendicular to the y-axis are squares.



$$V = \int_{-1}^{1} \left(1 - y^2\right)^2 dy$$

$$= \int_{-1}^{1} \left(1 - 2y^2 + y^4\right) dy$$

$$= \left(y - \frac{2y^3}{3} + \frac{y^5}{5}\right)\Bigg|_{-1}^{1}$$

$$= 1 - \frac{2}{3} + \frac{1}{5} - \left(-1 + \frac{2}{3} - \frac{1}{5}\right)$$

$$= 2 - \frac{4}{3} + \frac{2}{5} = \frac{30 - 20 + 6}{15} = \frac{16}{15}$$