Skill Builder: Topics 8.9-8.12 – Volumes Using Disc and Washer Methods

For each problem, sketch the region bounded by the graphs of the functions and find the area of the region. <u>Be sure to draw in all representative rectangles</u>. Problems marked with an calculator icon indicate that you may use your TI-Nspire to evaluate the definite integral.

1.) Find the volume if the region enclosing $y=4-x,\ x=0,$ and $y=0$ is rotated about the given line. Circle which method you will use and fill in the appropriate values of ***R*** and ***r*** where applicable.

a.) the *x*-axis

Disk Method

r = $4-x-0$

Washer Method

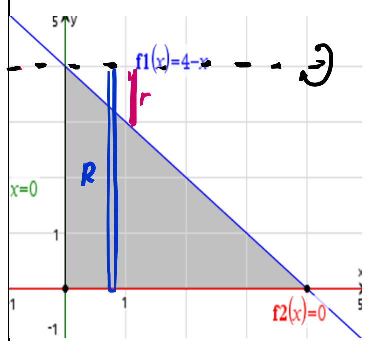
R = ______

r = ______

$$V=\pi\int_0^4(4-x)^2\,dx=\pi\int_0^4\left(16-8x+x^2\right)dx$$

$$=\pi\left(16x-4x^2+\frac{x^3}{3}\right)\Big|_0^4=\pi\left(64-64+\frac{64}{3}\right)=\frac{64}{3}\pi$$

b.) the line $y=4$



Disk Method

r = ______

Washer Method

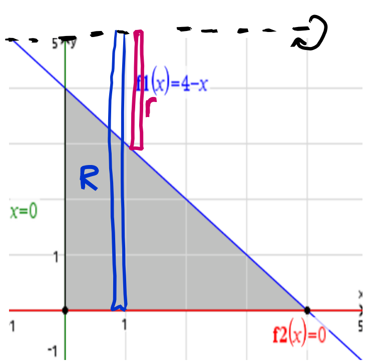
R = $4-0$

r = $4-(4-x)$

$$V=\pi\int_0^4\left(4^2-x^2\right)dx=\pi\int_0^4\left(16-x^2\right)dx$$

$$=\pi\left(16x-\frac{x^3}{3}\right)\Big|_0^4=\pi\left(64-\frac{64}{3}\right)=\frac{128}{3}\pi$$

c.) the line $y=5$



Disk Method

r = ______

Washer Method

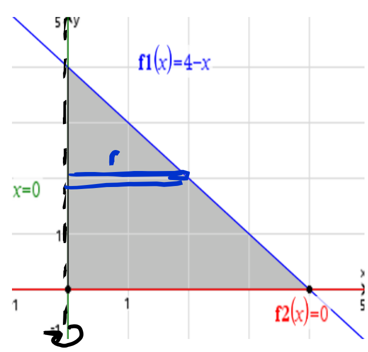
R = 5

r = $5-(4-x)$

$$V=\pi\int_0^4\left(5^2-(1+x)^2\right)dx=\pi\int_0^4\left(25-1-2x-x^2\right)dx=\pi\int_0^4\left(24-2x-x^2\right)dx$$

$$=\pi\left(24x-x^2-\frac{x^3}{3}\right)\Big|_0^4=\pi\left(96-16-\frac{64}{3}\right)=\frac{176}{3}\pi$$

d.) the *y*-axis



Disk Method

r = $4-y-0$

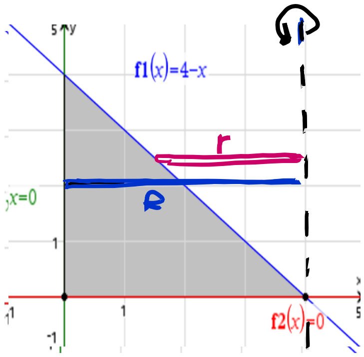
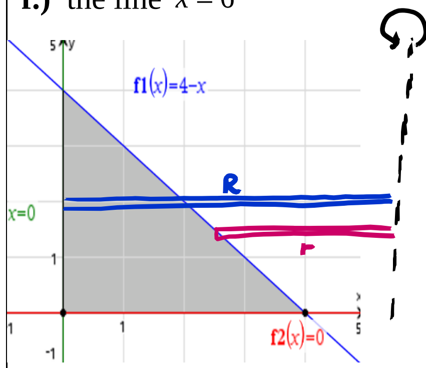
Washer Method

R = ______

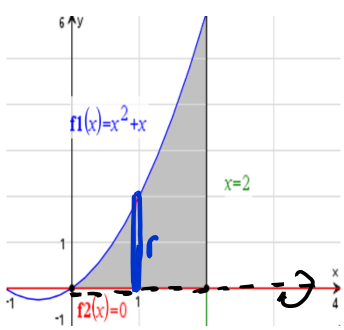
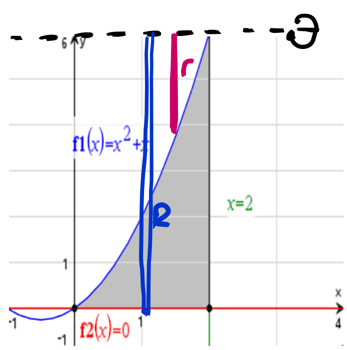
r = ______

$$V=\pi\int_0^4(4-y)^2\,dy=\pi\int_0^4\left(16-8y+y^2\right)dy$$

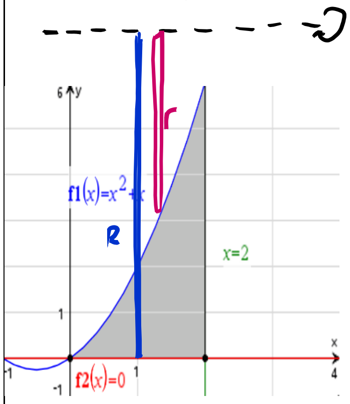
$$=\pi\left(16y-4y^2+\frac{y^3}{3}\right)\Big|_0^4=\pi\left(64-64+\frac{64}{3}\right)=\frac{64}{3}\pi$$

e.) the line $x = 4$	**f.)** the line $x = 6$
 **Disk Method** *r* = ____ **Washer Method** ***R*** = $4-0$ ***r*** = $4-(4-y)$	 **Disk Method** *r* = ____ **Washer Method** ***R*** = $6-0$ ***r*** = $6-(4-y)=2+y$
$V=\pi\int_0^4\left(4^2-y^2\right)dy=\pi\int_0^4\left(16-y^2\right)dy$ $=\pi\left(16y-\frac{y^3}{3}\right)\Big\|_0^4=\pi\left(64-\frac{64}{3}\right)=\frac{128}{3}\pi$	$V=\pi\int_0^4\left(6^2-(2+y)^2\right)dy=\pi\int_0^4\left(36-4-4y-y^2\right)dy$ $=\pi\left(32y-2y^2-\frac{y^3}{3}\right)\Big\|_0^4=\pi\left(128-32-\frac{64}{3}\right)=\frac{224}{3}\pi$

2.) Find the volume if the region enclosing $y = x^2 + x$, $y = 0$, and $x = 2$ is rotated about the given line. Circle which method you will use and fill in the appropriate values of ***R*** and ***r*** where applicable.

a.) the *x*-axis	**b.)** the line $y = 6$
 **Disk Method** ***r*** = x^2+x **Washer Method** ***R*** = ____ ***r*** = ____	 **Disk Method** *r* = ____ **Washer Method** ***R*** = $6-0$ ***r*** = $6-(x^2+x)$
$V=\pi\int_0^2\left(x^2+x\right)^2dx=\pi\int_0^2\left(x^4+2x^3+x^2\right)dx$ $=\pi\left(\frac{x^5}{5}+\frac{x^4}{2}+\frac{x^3}{3}\right)\Big\|_0^2=\pi\left(\frac{32}{5}+8+\frac{8}{3}\right)=\frac{256}{15}\pi$	$V=\pi\int_0^2\left((6-0)^2-\left(6-(x^2+x)\right)^2\right)dx=\pi\int_0^2\left(36-\left(6-x^2-x\right)^2\right)dx=$ $=\pi\int_0^2\left(-x^4-2x^3+11x^2+12x\right)dx$ $=\pi\left(-\frac{x^5}{5}-\frac{x^4}{2}+\frac{11x^3}{3}+6x^2\right)\Big\|_0^2=\frac{584}{15}\pi$

c.) the line $y=9$



~~Disk Method~~

~~$r=$ ______~~

Washer Method

$R=9-0$

$r=9-(x^2+x)$

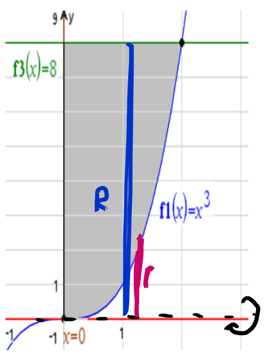
$$V=\pi\int_0^2\left((9-0)^2-\left(9-(x^2+x)\right)^2\right)dx=\pi\int_0^2\left(81-\left(9-x^2-x\right)^2\right)dx$$

$$=\pi\int_0^2\left(-x^4-2x^3+17x^2+18x\right)dx$$

$$=\pi\left(-\frac{x^5}{5}-\frac{x^4}{2}+\frac{17x^3}{3}+9x^2\right)\Bigg|_0^2=\frac{1004}{15}\pi$$

3.) Find the volume if the region enclosing $y=x^3$, $x=0$, and $y=8$ is rotated about the given line. Circle which method you will use and fill in the appropriate values of ***R*** and ***r*** where applicable.

a.) the x-axis



~~Disk Method~~

~~$r=$ ______~~

Washer Method

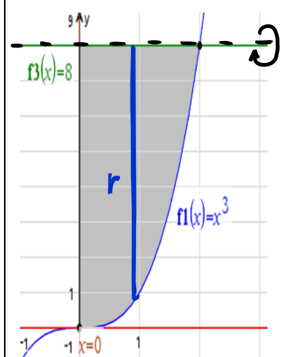
$R=8-0$

$r=x^3-0$

$$V=\pi\int_0^2\left((8-0)^2-\left(x^3-0\right)^2\right)dx=\pi\int_0^2\left(64-x^6\right)dx$$

$$=\pi\left(64x-\frac{x^7}{7}\right)\Bigg|_0^2=\pi\left(128-\frac{128}{7}\right)=\frac{768\pi}{7}$$

b.) the line $y=8$



Disk Method

$r=8-x^3$

~~Washer Method~~

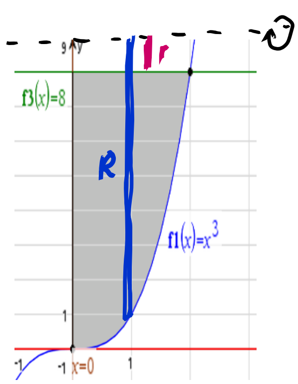
~~$R=$ ______~~

~~$r=$ ______~~

$$V=\pi\int_0^2\left(8-x^3\right)^2dx=\pi\int_0^2\left(64-16x^3+x^6\right)dx$$

$$=\pi\left(64x-4x^4+\frac{x^7}{7}\right)\Bigg|_0^2=\pi\left(128+64+\frac{128}{7}\right)=\frac{576}{7}\pi$$

c.) the line $y=9$



~~Disk Method~~

~~$r=$ ______~~

Washer Method

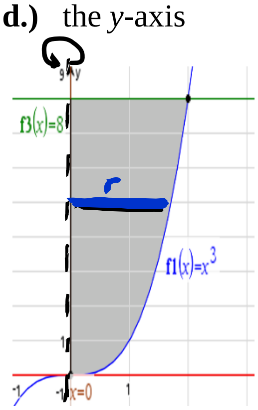
$R=9-x^3$

$r=9-8$

$$V=\pi\int_0^2\left((9-x^3)^2-(9-8)^2\right)dx=\pi\int_0^2\left(80-18x^3+x^6\right)dx$$

$$=\pi\left(80x-\frac{9x^4}{2}+\frac{x^7}{7}\right)\Bigg|_0^2=\pi\left(160-72+\frac{128}{7}\right)=\frac{744\pi}{7}$$

d.) the y-axis



Disk Method

$r=\sqrt[3]{y}-0$

~~Washer Method~~

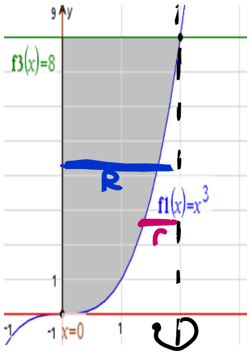
~~$R=$ ______~~

~~$r=$ ______~~

$$V=\pi\int_0^8\left(y^{1/3}-0\right)^2dy=\pi\int_0^8\left(y^{2/3}\right)dy$$

$$=\pi\left(\frac{3}{5}y^{5/3}\right)\Bigg|_0^8=\pi\left(\frac{3}{5}\cdot 32\right)=\frac{96}{5}\pi$$

e.) the line $x = 2$



~~Disk Method~~

~~$r =$ ______~~

Washer Method

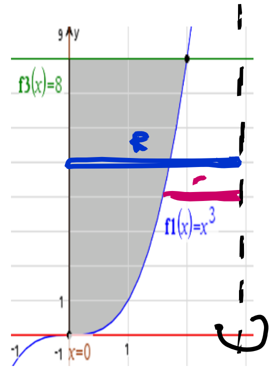
$R = 2 - 0$

$r = 2 - \sqrt[3]{y}$

$$V = \pi\int_0^8 \left((2-0)^2 - \left(2 - y^{1/3}\right)^2\right)dy = \pi\int_0^8 \left(4 - 4 + 4y^{1/3} - y^{2/3}\right)dx$$

$$= \pi\left(4\cdot\frac{3}{4}y^{4/3} - \frac{3}{5}y^{5/3}\right)\Big|_0^8 = \pi\left(48 - \frac{96}{5}\right) = \frac{144\pi}{5}$$

f.) the line $x = 3$



~~Disk Method~~

~~$r =$ ______~~

Washer Method

$R = 3 - 0$

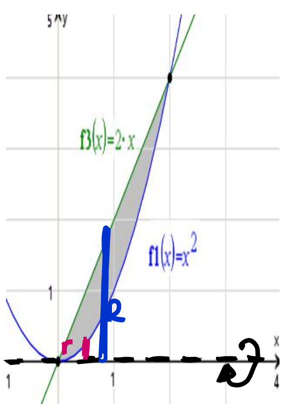
$r = 3 - \sqrt[3]{y}$

$$V = \pi\int_0^8 \left((3-0)^2 - \left(3 - y^{1/3}\right)^2\right)dy = \pi\int_0^8 \left(9 - 9 + 6y^{1/3} - y^{2/3}\right)dx$$

$$= \pi\left(6\cdot\frac{3}{4}y^{4/3} - \frac{3}{5}y^{5/3}\right)\Big|_0^8 = \pi\left(72 - \frac{96}{5}\right) = \frac{264\pi}{5}$$

4.) Find the volume if the region enclosing $y = x^2$ and $y = 2x$ where $x \geq 0$ is rotated about the given line. Circle which method you will use and fill in the appropriate values of ***R*** and ***r*** where applicable.

a.) the *x*-axis



~~Disk Method~~

~~$r =$ ______~~

Washer Method

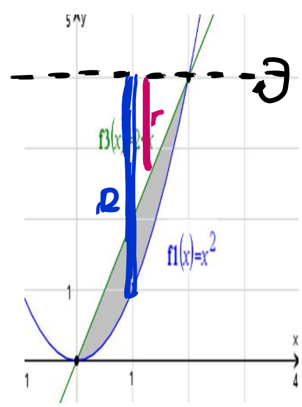
$R = 2x - 0$

$r = x^2 - 0$

$$V = \pi\int_0^2 \left((2x-0)^2 - \left(x^2 - 0\right)^2\right)dx = \pi\int_0^2 \left(4x^2 - x^4\right)dx$$

$$= \pi\left(4\cdot\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^2 = \pi\left(\frac{32}{3} - \frac{32}{5}\right) = \frac{64\pi}{15}$$

b.) the line $y = 4$



~~Disk Method~~

~~$r =$ ______~~

Washer Method

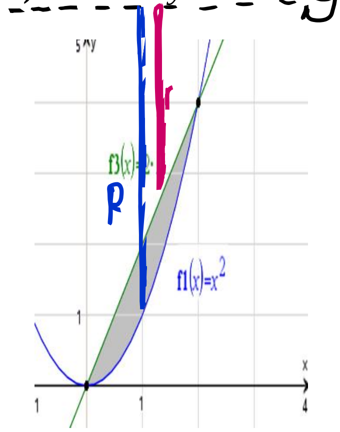
$R = 4 - x^2$

$r = 4 - 2x$

$$V = \pi\int_0^2 \left((4-x^2)^2 - \left(4 - 2x\right)^2\right)dx =$$

$$= \frac{32\pi}{5}$$

c.) the line $y = 7$



Disk Method

r = ______

Washer Method

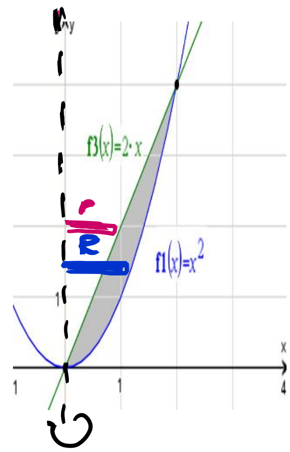
R = $7 - x^2$

r = $7 - 2x$

$$V = \pi\int_0^2 \left((7 - x^2)^2 - (7 - 2x)^2\right)dx =$$

$$= \frac{72\pi}{5}$$

d.) the *y*-axis



Disk Method

r = ______

Washer Method

R = $\sqrt{y} - 0$

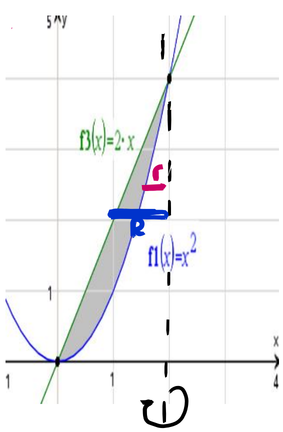
r = $\frac{y}{2} - 0$

$$y = 2x \rightarrow x = \frac{y}{2}; \quad y = x^2 \rightarrow x = \sqrt{y}$$

$$V = \pi\int_0^4 \left(\left(\sqrt{y} - 0\right)^2 - \left(\frac{y}{2} - 0\right)^2\right)dy$$

$$= \frac{8\pi}{3}$$

e.) the line $x = 2$



Disk Method

r = ______

Washer Method

R = $2 - \frac{y}{2}$

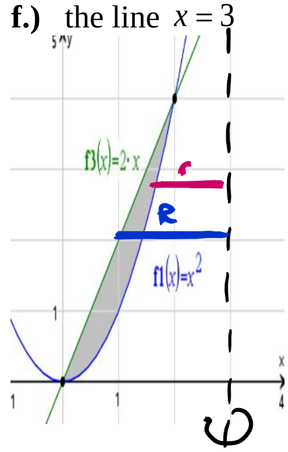
r = $2 - \sqrt{y}$

$$y = 2x \rightarrow x = \frac{y}{2}; \quad y = x^2 \rightarrow x = \sqrt{y}$$

$$V = \pi\int_0^4 \left(\left(2 - \frac{y}{2}\right)^2 - \left(2 - \sqrt{y}\right)^2\right)dy$$

$$= \frac{8\pi}{3}$$

f.) the line $x = 3$



Disk Method

r = ______

Washer Method

R = $3 - \frac{y}{2}$

r = $3 - \sqrt{y}$

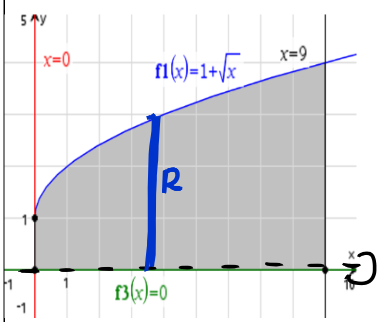
$$y = 2x \rightarrow x = \frac{y}{2}; \quad y = x^2 \rightarrow x = \sqrt{y}$$

$$V = \pi\int_0^4 \left(\left(3 - \frac{y}{2}\right)^2 - \left(3 - \sqrt{y}\right)^2\right)dy$$

$$= \frac{16\pi}{3}$$

5.) Find the volume if the region enclosing $y = 1 + \sqrt{x}$, $x = 0$, $y = 0$ and $x = 9$ is rotated about the given line. Circle which method you will use and fill in the appropriate values of ***R*** and ***r*** where applicable.

a.) the x-axis



Disk Method

$r = 1 + \sqrt{x}$

Washer Method

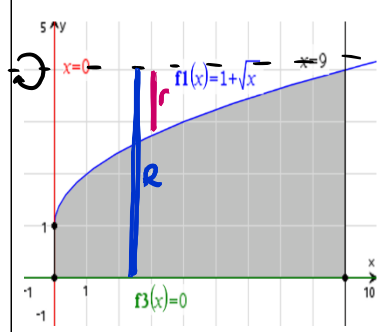
R = ________

r = ________

$$V = \pi \int_0^9 \left(1 + \sqrt{x}\right)^2 dx$$

$$= \frac{171\pi}{2}$$

b.) the line $y = 4$



Disk Method

r = ________

Washer Method

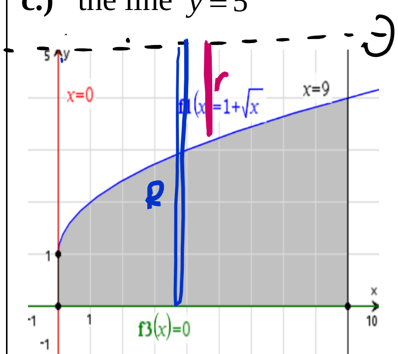
$R = 4 - 0$

$r = 4 - \left(1 + \sqrt{x}\right)$

$$V = \pi \int_0^9 \left[(4-0)^2 - \left(4 - \left(1 + \sqrt{x}\right)\right)^2 \right] dx$$

$$= \frac{261\pi}{2}$$

c.) the line $y = 5$



Disk Method

r = ________

Washer Method

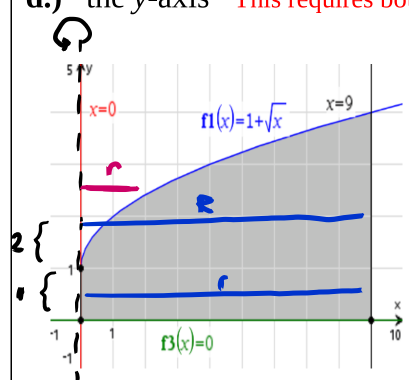
$R = 5 - 0$

$r = 5 - \left(1 + \sqrt{x}\right)$

$$V = \pi \int_0^9 \left[(5-0)^2 - \left(5 - \left(1 + \sqrt{x}\right)\right)^2 \right] dx$$

$$= \frac{369\pi}{2}$$

d.) the y-axis — This requires both disk and washer methods.



Disk Method

$r = 9 - 0$

Washer Method

$R = 9 - 0$

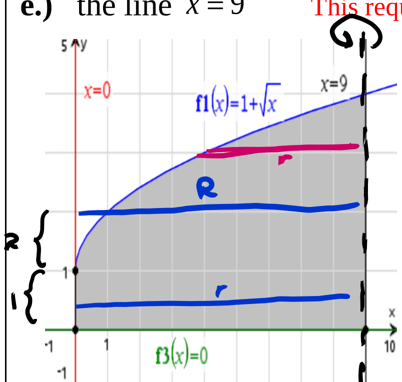
$r = (y-1)^2 - 0$

$$y = 1 + \sqrt{x} \rightarrow \sqrt{x} = y - 1 \rightarrow x = (y-1)^2$$

$$V = V_1 + V_2 = \pi \int_0^1 (9-0)^2\, dy + \pi \int_1^4 \left[(9-0)^2 - \left((y-1)^2 - 0\right)^2 \right] dy$$

$$= \frac{1377\pi}{5}$$

e.) the line $x = 9$ — This requires two disk methods.



Disk Method

$r_1 = 9 - 0$

$r_2 = 9 - (y-1)^2$

Washer Method

R = ________

r = ________

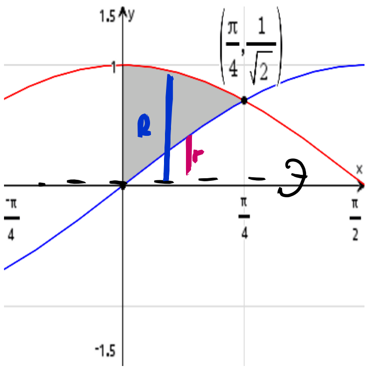
$$y = 1 + \sqrt{x} \rightarrow \sqrt{x} = y - 1 \rightarrow x = (y-1)^2$$

$$V = V_1 + V_2 = \pi \int_0^1 (9-0)^2\, dy + \pi \int_1^4 \left(9 - (y-1)^2\right)^2 dy$$

$$= \frac{1053\pi}{5}$$

6.) Find the volume if the first quadrant region enclosing $y = \sin x$ and $y = \cos x$ on $\left[0, \frac{\pi}{4}\right]$ is rotated about the given line. Circle which method you will use and fill in the appropriate values of ***R*** and ***r*** where applicable.

a.) the *x*-axis



Disk Method

r = ________

Washer Method

R = $\cos x - 0$

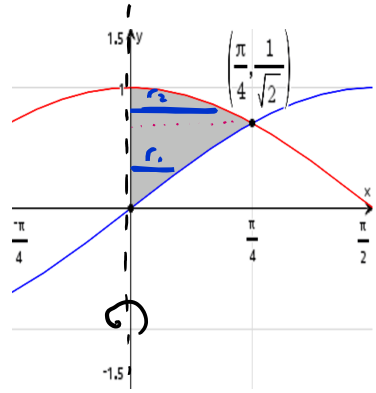
r = $\sin x - 0$

$$V = \pi \int_0^{\pi/4} \left[(\cos x - 0)^2 - (\sin x - 0)^2 \right] dx$$

$$= \frac{\pi}{2}$$

Note: $\cos^2 x - \sin^2 x = \cos 2x$. This would be fairly easy to integrate.

b.) the *y*-axis This requires two disk methods.



Disk Method

r₁ = $\arcsin y - 0$

r₂ = $\arccos y - 0$

Washer Method

R = ________

r = ________

$$V = V_1 + V_2 = \pi \int_0^{1/\sqrt{2}} (\arcsin y - 0)^2 \, dy + \pi \int_{1/\sqrt{2}}^{1} (\arccos y - 0)^2 \, dy$$

$$\approx 0.221\pi$$

7.) A tank on the wing of a jet plane is formed by revolving the region bounded by the graph of $y = \frac{1}{10} x^2 \sqrt{3 - x}$ and the *x*-axis about the *x*-axis where *x* and *y* are measured in meters. Find the volume of the tank. A calculator may be needed to simplify the definite integral's result.

$$V = \pi \int_0^3 \left(\frac{1}{10} x^2 \sqrt{3 - x} \right)^2 dx$$

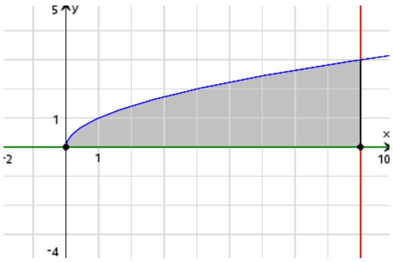
$$= \frac{\pi}{100} \int_0^3 \left(x^4 (3 - x) \right) dx = \frac{\pi}{100} \int_0^3 \left(3x^4 - x^5 \right) dx$$

$$= \frac{\pi}{100} \left(\frac{3x^5}{5} - \frac{x^6}{6} \right) \Bigg|_0^3$$

$$= \frac{\pi}{100} \left(\frac{729}{5} - \frac{2187}{6} \right)$$

$$= \frac{243\pi}{1000}$$

8.) The region bounded by the curve $y=\sqrt{x},\ x=0,\ y=0$ and $x=9$ is rotated about the x-axis.



a.) Find the value of a in the interval [0,9] that divides the region into 2 parts of equal area.
Write your answer as both an exact value and as a decimal approximation using your calculator.

$$A=\int_0^9\left(\sqrt{x}\right)dx=\frac{2}{3}x^{3/2}\Big|_0^9=\frac{2}{3}\cdot 27=18$$

$$\int_0^a\left(\sqrt{x}\right)dx=9$$

$$\frac{2}{3}x^{3/2}\Big|_0^a=9$$

$$\frac{2}{3}a^{3/2}=9$$

$$a^{3/2}=\frac{27}{2}$$

$$a=\left(\frac{27}{2}\right)^{2/3}\approx 5.6696$$

b.) Find the value of a in the interval [0,9] that divides the region into 2 parts of equal volume.
Write your answer as both an exact value and as a decimal approximation using your calculator.

$$V=\pi\int_0^9\left(\sqrt{x}\right)^2 dx=\pi\int_0^9 x\,dx=\pi\cdot\frac{x^2}{2}\Big|_0^9=\frac{81\pi}{2}$$

$$\pi\int_0^a\left(\sqrt{x}\right)^2 dx=\frac{81\pi}{4}$$

$$\frac{\pi x^2}{2}\Big|_0^a=\frac{81\pi}{4}$$

$$\frac{\pi a^2}{2}=\frac{81\pi}{4}$$

$$a^2=\frac{81}{2}$$

$$a=\frac{9}{\sqrt{2}}\approx 6.3640$$