Name ___________________________

AP Calculus AB
Unit 3 Review
DIFFERENTIATION – Compositions, Implicit and Inverse Functions

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1.) Find $\frac{dy}{dx}$ for $y = x^3\sqrt{x+1}$ and simplify.

$y = x^3(x+1)^{1/2}$

Use the Product Rule

$\frac{dy}{dx} = 3x^2(x+1)^{1/2} + x^3 \cdot \frac{1}{2}(x+1)^{-1/2}$

Simplify by factoring out the common factors.

$\frac{dy}{dx} = x^2(x+1)^{-1/2}\left[3(x+1) + \frac{1}{2}x\right]$

$\frac{dy}{dx} = \frac{x^2\left(3x+3+\frac{1}{2}x\right)}{\sqrt{x+1}} = \frac{x^2\left(\frac{7}{2}x+3\right)}{\sqrt{x+1}}$

$\frac{dy}{dx} = \frac{x^2(7x+6)}{2\sqrt{x+1}}$

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2.) Find $f'(x)$ if $f(x) = \cos^5(4x)$

$f(x) = [\cos(4x)]^5$

$f'(x) = 5[\cos(4x)]^4 \cdot \frac{d}{dx}[\cos(4x)]$

$f'(x) = 5\cos^4(4x) \cdot (-\sin(4x)) \cdot 4$

$f'(x) = -20\cos^4(4x)\sin(4x)$

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3.) Find $f'(\theta)$ if $f(\theta) = \sqrt{\sin(2\theta)}$

$f(\theta) = [\sin(2\theta)]^{1/2}$

$f'(\theta) = \frac{1}{2}[\sin(2\theta)]^{-1/2} \cdot \frac{d}{d\theta}[\sin(2\theta)]$

$f'(\theta) = \frac{1}{2}[\sin(2\theta)]^{-1/2} \cdot \cos(2\theta) \cdot 2$

$f'(\theta) = \frac{\cos(2\theta)}{\sqrt{\sin(2\theta)}}$

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4.) Given $f(x) = x^2(2x-5)^2$, find $f'(x)$ and simplify.

$f(x) = x^2(2x-5)^2$

Use the Product Rule

$f'(x) = 2x(2x-5)^2 + x^2 \cdot 2(2x-5)^1 \cdot 2$

Simplify by factoring out the common factors.

$f'(x) = 2x(2x-5)[(2x-5)+2x]$

$f'(x) = 2x(2x-5)(4x-5)$

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5.) Find $\dfrac{d^2y}{dx^2}$ for $x^2+xy+y^2=3$

$$\frac{d}{dx}\left[x^2+xy+y^2\right]=\frac{d}{dx}[3]$$

$$2x+1\cdot y+x\cdot\frac{dy}{dx}+2y\cdot\frac{dy}{dx}=0$$

$$x\cdot\frac{dy}{dx}+2y\cdot\frac{dy}{dx}=-2x-y$$

$$\frac{dy}{dx}(x+2y)=-2x-y$$

$$\frac{dy}{dx}=\frac{-2x-y}{x+2y}$$

Use the Quotient Rule

$$\frac{d^2y}{dx^2}=\frac{\left(-2-\frac{dy}{dx}\right)(x+2y)-(-2x-y)\left(1+2\frac{dy}{dx}\right)}{(x+2y)^2}$$

Replace $\frac{dy}{dx}$.

$$\frac{d^2y}{dx^2}=\frac{\left(-2-\frac{-2x-y}{x+2y}\right)(x+2y)-(-2x-y)\left(1+2\cdot\frac{-2x-y}{x+2y}\right)}{(x+2y)^2}$$

We are not going to simplify any further.

6.) Find y'' for $y=\dfrac{\csc x}{2}$

$$y=\frac{1}{2}\csc x$$

$$y'=\frac{1}{2}(-\csc x\cot x)=-\frac{1}{2}(\csc x\cot x)$$

$$y''=-\frac{1}{2}\left(-\csc x\cot x\cdot\cot x+(\csc x)\cdot(-\csc^2 x)\right)$$

$$y''=\frac{1}{2}\csc x\left(\cot^2 x+\csc^2 x\right)$$

7.) Find y' for $y = \ln\sqrt{x^2 - 4x - 7}$

Rewrite using logarithmic rules.

$$y = \frac{1}{2}\ln\left(x^2 - 4x - 7\right)$$

$$y' = \frac{1}{2}\cdot\frac{2x-4}{x^2-4x-7}$$

$$y' = \frac{x-2}{x^2-4x-7}$$

8.) Find $h'(x)$ for $h(x) = \ln\frac{x(x-1)^3}{\sqrt{x-2}}$. You do not have to simplify.

Rewrite using logarithm rules.

$$h(x) = \ln x + 3\ln(x-1) - \frac{1}{2}\ln(x-2)$$

$$h'(x) = \frac{1}{x} + \frac{3}{x-1} - \frac{1}{2(x-2)}$$

9.) Find y' for $y = -4e^{\sec x}$

$$y' = -4e^{\sec}\cdot\frac{d}{dx}[\sec x]$$

$$y' = -4e^{\sec x}\cdot\sec x\tan x$$

10.) Find y' for $y = 3^{5x}$

$$y' = (\ln 3)\cdot 3^{5x}\cdot\frac{d}{dx}[5x]$$

$$y' = 5(\ln 3)\cdot 3^{5x}$$

11.) Suppose that $h(x) = f(g(x))$ and $g(14) = 2,\ g'(14) = 5,\ f'(14) = 15$ and $f'(2) = 12$. Find $h'(14)$.

$$h'(x) = f'(g(x))\cdot g'(x)$$

$$h'(14) = f'(g(14))\cdot g'(14)$$

$$h'(14) = f'(2)\cdot 5$$

$$h'(14) = 12\cdot 5 = 60$$

12.) Find $\frac{dy}{dx}$ for $y = x^4\sqrt{x+1}$.

(A) $\frac{dy}{dx} = \frac{x^3(9x+8)}{2\sqrt{x+1}}$

(B) $\frac{dy}{dx} = \frac{4x^3}{2\sqrt{x+1}}$

(C) $\frac{dy}{dx} = \frac{x^3(9x+1)}{2\sqrt{x+1}}$

(D) $\frac{dy}{dx} = 4x^3\sqrt{x+1}$

Rewrite

$$y = x^4(x+1)^{1/2}$$

Use the Product Rule

$$y' = 4x^3(x+1)^{1/2} + x^4\cdot\frac{1}{2}(x+1)^{-1/2}$$

Factor out a comon factor.

$$y' = \frac{1}{2}x^3(x+1)^{-1/2}[8(x+1)+x]$$

$$y' = \frac{x^3(9x+8)}{2\sqrt{x+1}}$$

13.) Suppose that $f(x)$ and $g(x)$ and their derivatives have the following values at $x=-1$ and $x=0$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	0	-1	2	1
0	-1	-3	-2	4

Evaluate the first derivative of each of the following combinations of $f(x)$ and $g(x)$ at the given value of x.

a.) $g(f(x)),\quad x=-1$

$g'(f(x))\cdot f'(x)$

$g'(f(-1))\cdot f'(-1)$

$g'(0)\cdot 2$

$4\cdot 2=8$

b.) $f(g(x)),\quad x=-1$

$f'(g(x))\cdot g'(x)$

$f'(g(-1))\cdot g'(-1)$

$f'(-1)\cdot 1$

$2\cdot 1=2$

c.) $g(x+f(x)),\quad x=0$

$g'(x+f(x))\cdot(1+f'(x))$

$g'(0+f(0))\cdot(1+f'(0))$

$g'(0+(-1))\cdot(1+(-2))$

$g'(-1)\cdot(-1)$

$1\cdot(-1)=-1$

14.) Find $f''(x)$ if $f(x) = \tan 5x + x^2$.

(A) $f''(x) = \sec^2 5x + 2x$

(B) $f''(x) = 50\sec 5x \cdot \tan 5x + 2x$

(C) $f''(x) = 10\sec^2 5x \cdot \tan 5x + 2$

(D) $f''(x) = 50\sec^2 5x \cdot \tan 5x + 2$

$f'(x) = \sec^2(5x) \cdot 5 + 2x$ or $5[\sec(5x)]^2 + 2x$

$f''(x) = 10\sec(5x) \cdot \frac{d}{dx}[\sec(5x)] + 2$

$f''(x) = 10\sec(5x) \cdot \sec(5x)\tan(5x) \cdot 5 + 2$

$f''(x) = 50\sec^2(5x)\tan(5x) + 2$

15.) The functions f and g are differentiable for all real numbers x. The table below gives values of the function and their first derivatives at selected values of x with a being a constant.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	$\frac{3\pi}{4}$	$\sqrt{2}$	0	$\frac{1}{a}$
2	8	-6	3	-4
3	1	5	4	$\frac{1}{2}$
4	5	$2a^2$	9	3

a.) If $h(x) = \sin(f(x))$, write an equation of the line tangent to h at the point where $x = 1$.

$h'(x) = \cos(f(x)) \cdot f'(x)$

$h'(1) = \cos(f(1)) \cdot f'(1)$

$h'(1) = \cos\left(\frac{3p}{4}\right) \cdot \sqrt{2}$

$h'(1) = -\frac{1}{\sqrt{2}} \cdot \sqrt{2} = -1$

$h(1) = \sin(f(1)) = \sin\left(\frac{3p}{4}\right) = \frac{1}{\sqrt{2}}$

$y - \frac{1}{\sqrt{2}} = -(x - 1)$

b.) If $r(x)=\dfrac{1}{\sqrt{g(2x)}}$, find $r'(x)$ at $x=2$.	**c.)** Find the value(s) of *a* if the tangent lines to $f\left(g(x)\right)$ and $g\left(f(x)\right)$ are perpendicular at $x=3$.

b.) If $r(x)=\dfrac{1}{\sqrt{g(2x)}}$, find $r'(x)$ at $x=2$.

$$r(x)=\left(g(2x)\right)^{-1/2}$$

$$r'(x)=-\frac{1}{2}\left(g(2x)\right)^{-3/2}\cdot\frac{d}{dx}\left[g(2x)\right]$$

$$=-\frac{1}{2}\left(g(2x)\right)^{-3/2}\cdot g'(2x)\cdot 2$$

$$r'(2)=-\frac{1}{2}\left(g(4)\right)^{-3/2}\cdot g'(4)\cdot 2$$

$$=-\frac{1}{2}(9)^{-3/2}\cdot(3\cdot 2)=-\frac{6}{2\cdot 9^{3/2}}$$

$$=-\frac{6}{2\cdot 27}=-\frac{1}{9}$$

c.) Find the value(s) of *a* if the tangent lines to $f\left(g(x)\right)$ and $g\left(f(x)\right)$ are perpendicular at $x=3$.

$$f'\left(g(x)\right)\cdot g'(x) \qquad g'\left(f(x)\right)\cdot f'(x)$$

$$f'\left(g(3)\right)\cdot g'(3) \qquad g'\left(f(3)\right)\cdot f'(3)$$

$$f'(4)\cdot\frac{1}{2} \qquad g'\left(1\right)\cdot 5$$

$$2a^2\cdot\frac{1}{2} \qquad \frac{1}{a}\cdot 5$$

$$a^2 \qquad \frac{5}{a}$$

To be perpendicular, $a^2=-\dfrac{a}{5}$

$$5a^2=-a \Rightarrow 5a^2+a=0 \Rightarrow a(5a+1)=0$$

$$a=0,\ a=-\frac{1}{5}$$

16.) Find $\dfrac{dy}{dx}$ if $ye^x-x=y^2$.

$$\frac{dy}{dx}\cdot e^x+ye^x-1=2y\frac{dy}{dx}$$

$$\frac{dy}{dx}e^x-2y\frac{dy}{dx}=1-ye^x$$

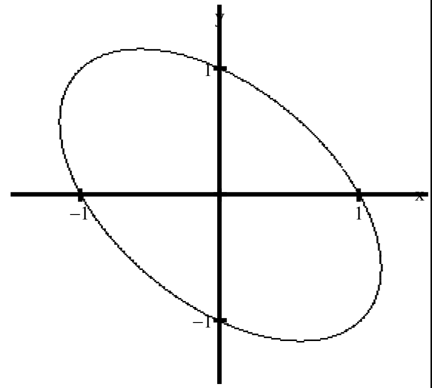
$$\frac{dy}{dx}\left(e^x-2y\right)=1-ye^x$$

$$\frac{dy}{dx}=\frac{1-ye^x}{e^x-2y}$$

17.) Consider the curve given by $2x^2 + y^2 - xy = 4$.
a.) Find $\frac{dy}{dx}$. $4x + 2y\frac{dy}{dx} - \left(1y + x\frac{dy}{dx}\right) = 0$ $4x + 2y \cdot \frac{dy}{dx} - y - x \cdot \frac{dy}{dx} = 0$ $2y\frac{dy}{dx} - x\frac{dy}{dx} = y - 4x$ $\frac{dy}{dx}(2y - x) = y - 4x$ $\frac{dy}{dx} = \frac{y-4x}{2y-x}$
b.) Write the equations of the tangent lines when $x = 1$. When $x = 1$, $2(1)^2 + y^2 - (1)y = 4$ $2 + y^2 - y = 4$ $y^2 - y - 2 = 0 \quad \to \quad (y-2)(y+1) = 0 \quad \to \quad y = 2,\ y = -1$ $\left.\frac{dy}{dx}\right\|_{(1,2)} = \frac{2-4}{2(2)-1} = \frac{-2}{3}$ $\quad \left.\frac{dy}{dx}\right\|_{(1,-1)} = \frac{-1-4}{2(-1)-1} = \frac{-5}{-3} = \frac{5}{3}$ $y - 2 = -\frac{2}{3}(x-1)$ $\quad y - (-1) = \frac{5}{3}(x-1)$
c.) Find the y-values of each point on the curve $2x^2 + y^2 - xy = 4$ where the tangent line is vertical. Vertical tangent lines occur when the derivative is undefined. In this case, this occurs when the denominator of $\frac{dy}{dx}$ is equal to zero. $2y - x = 0 \ \to x = 2y$ Substitute $x = 2y$ into the original equation to solve. $2(2y)^2 + y^2 - (2y)y = 4 \to 8y^2 + y^2 - 2y^2 = 4 \to 7y^2 = 4$ $y^2 = \frac{4}{7} \to \quad y = \sqrt{\frac{4}{7}},\ -\sqrt{\frac{4}{7}}$

18.) Consider the graph of $x^2 + xy + y^2 = 1$.

a.) Find an expression for $\frac{dy}{dx}$ in terms of x and y.



$$2x + 1 \cdot y + x \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = 0$$

$$x \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx}(x + 2y) = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x - y}{x + 2y}$$

b.) Find all points on the curve where the tangent line is parallel to the line $y = -2x$.

$$\frac{-2x - y}{x + 2y} = -2 \quad \rightarrow \quad -2x - y = -2x - 4y$$

Fortunately the x terms cancel away.

$$-y = -4y \quad \rightarrow \quad 3y = 0 \quad \rightarrow \quad y = 0$$

Substitute $y = 0$ into the original equation.

$$x^2 + 0x + 0^2 = 1 \quad \rightarrow \quad x^2 = 1 \quad \rightarrow \quad x = 1, x = -1$$

The points are $(1, 0)$ and $(-1, 0)$.

c.) Find all x-values on the curve where the tangent is horizontal.

Horizontal tangent lines occur when $\frac{dy}{dx} = 0$.

Set the numerator of the derivative equal to 0.

$$-2x - y = 0 \quad \rightarrow \quad y = -2x$$

Substitute $y = -2x$ into the original equation.

$$x^2 + x(-2x) + (-2x)^2 = 1$$

$$x^2 - 2x^2 + 4x^2 = 1$$

$$3x^2 = 1$$

$$x^2 = \frac{1}{3} \qquad \therefore \quad x = \frac{1}{\sqrt{3}}, \quad x = -\frac{1}{\sqrt{3}}$$

19.) Given:

$f(4)=6,\ f'(4)=7,\ f(6)=10,$

$f'(6)=-5$. What is $\left(f^{-1}\right)'(10)$?

$$\left(f^{-1}\right)'(10)=\frac{1}{f'\left(f^{-1}(10)\right)}$$

$$=\frac{1}{f'(6)}=\frac{1}{-5}=-\frac{1}{5}$$

20.) Find $\left(f^{-1}\right)'(-12)$ for the function $f(x)=\frac{1}{3}x^3+\frac{5}{3}x+2$.

$$\left(f^{-1}\right)'(-12)=\frac{1}{f'\left(f^{-1}(-12)\right)}$$

$$-12=\frac{1}{3}x^3+\frac{5}{3}x+2$$

By trial and error, $x=-3$ will solve this equation.

$$\left(f^{-1}\right)'(-12)=\frac{1}{f'(-3)}$$

$$f'(x)=x^2+\frac{5}{3}$$

$$f'(-3)=9+\frac{5}{3}=\frac{32}{3}$$

$$\left(f^{-1}\right)'(-12)=\frac{1}{\frac{32}{3}}=\frac{3}{32}$$

21.) Let $f(x)=\sqrt{e^x+3x+8}$ and let g be the inverse of the function f. Given $f(0)=3$, what is the value of $g'(3)$?

$$g'(3)=\left(f^{-1}\right)'(3)=\frac{1}{f'\left(f^{-1}(3)\right)}$$

$$3=\sqrt{e^x+3x+8}$$

$$9=e^x+3x+8$$

By trial and error, $x=0$ will solve this equation.

$$\left(f^{-1}\right)'(3)=\frac{1}{f'(0)}$$

$$f'(x)=\frac{1}{2}(e^x+3x+8)^{-1/2}\cdot(e^x+3)$$

$$f'(0)=\frac{e^0+3}{2\sqrt{e^0+3(0)+8}}=\frac{4}{2\sqrt{9}}=\frac{4}{6}=\frac{2}{3}$$

$$\left(f^{-1}\right)'(3)=\frac{1}{\frac{2}{3}}=\frac{3}{2}$$

22.) Find the derivative of the function $f(x)=\arccos\left(3x^2-1\right)$.

(A) $f'(x)=-\dfrac{6x}{\sqrt{1+\left(3x^2-1\right)^2}}$

(B) $f'(x)=-\dfrac{3x^2-1}{\sqrt{1-\left(3x^2-1\right)^2}}$

(C) $f'(x)=-\dfrac{6x}{\sqrt{\left(3x^2-1\right)^2-1}}$

(D) $f'(x)=-\dfrac{6x}{\sqrt{1-\left(3x^2-1\right)^2}}$

$u=3x^2-1 \quad u'=6x \quad u^2=(3x^2-1)^2$

$$f'(x)=\frac{-6x}{\sqrt{1-(3x^2-1)^2}}$$

23.) Find the derivative of the function $f(x) = x^2 \cdot \arctan(5x)$.

(A) $f'(x) = \dfrac{10x}{1+25x^2}$

(B) $f'(x) = \dfrac{10x^2}{1+25x^2}$

(C) $f'(x) = 2x\arctan(5x) + \dfrac{5x^2}{1+25x^2}$

(D) $f'(x) = 2x\arctan(5x) + \dfrac{5x^2}{1+5x^2}$

We must use the Product Rule.

$$f'(x) = 2x \cdot \arctan(5x) + x^2 \cdot \frac{5}{1+25x^2}$$

$$= 2x \cdot \arctan(5x) + \frac{5x^2}{1+25x^2}$$

24.) Let $f(x) = \arcsin x - 2x$.

a.) Find $f'(x)$.

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - 2$$

b.) Find the equation of the tangent line to $f(x)$ at the point where $x = 0$.

$$f'(0) = \frac{1}{\sqrt{1-(0)^2}} - 2 = 1 - 2 = -1$$

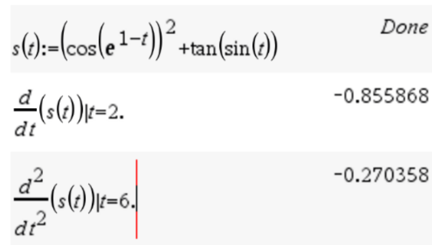
$$f(0) = \arcsin(0) - 2(0) = 0$$

$$y - 0 = -1(x - 0) \quad \text{or} \quad y = -x$$

25.) For each of following problems, use correct units when necessary and round according to the rules discussed in class.

Parts **a** and **b** refer to the following situation.
The position of a particle moving along the x-axis for any time, $t \geq 0$, is given by

$$s(t) = \left(\cos\left(e^{1-t}\right)\right)^2 + \tan\left(\sin t\right)$$ where $s(t)$ is measured in inches and t is measured in seconds.



a. Find the velocity of the particle at $t = 2$ seconds.

$v(2) = s'(2) = -0.856$ in/sec

b. Find the acceleration of the particle at $t = 6$ seconds.

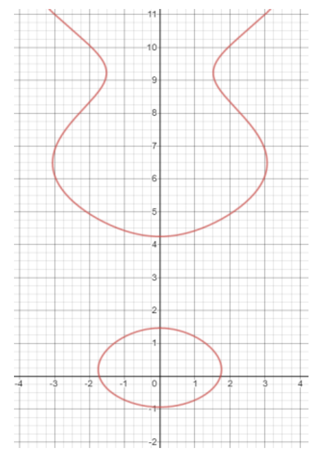
$a(6) = s''(6) = -0.270$ in/sec^2

Note: By entering a decimal point after the "2," the calculator output will be a decimal value rather than a complicated exact value in terms of various trigonometric expressions.

The second derivative can be accessed from the Math Template button.

Parts **c** and **d** refer to the following situation.

The graph of the relation $x^2 = -2 + y + 5\cos y$ for values of y such that $-1 \leq y \leq 11$ is shown to the right.



c. It is known that $\dfrac{dy}{dx} = \dfrac{2x}{1 - 5\sin y}$. Verify that this is true.

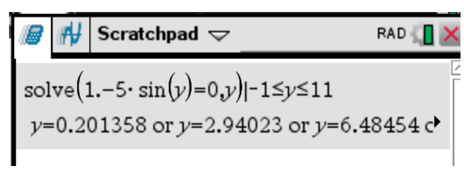
$$\frac{d}{dx}\left[x^2\right] = \frac{d}{dx}\left[-2 + y + 5\cos y\right]$$

$$2x = \frac{dy}{dx} - 5\sin y \cdot \frac{dy}{dx}$$

$$2x = \frac{dy}{dx}(1 - 5\sin y)$$

$$\frac{dy}{dx} = \frac{2x}{1 - 5\sin y}$$

d. For $y \leq 11$, the graph possesses three different vertical tangent lines. What is the equation of the vertical tangent line that has the smallest positive x-value.



Vertical tangents occur when the denomonator of the derivative is equal to zero.
In this case, $1 - 5\sin y = 0$. Use your calcualtor to solve this equation.
Be sure to include the restriction "$-1 \leq y \leq 11$" as well as a decimal point somewhere in your equation so that your output will be easy to interpret. Because there are so many solutions to this equation, you will have to scroll to the right using the Arrow Pad to see them all.
Eliminate answers that don't appear on the graph.

$y = 0.201,\ 6.485,\ 9.223$ The vertical tangent line with the smallest x value will occur when $y = 9.223$. This can be verified from the graph.

Lastly, we must solve the equation $x^2 = -2 + (9.223) + 5\cos(9.223)$

Thus, the equation of that vertical tangent line is $x = 1.525$