Avon High School
AP Calculus AB
UNIT 4 – Contextual Applications of Differentiation

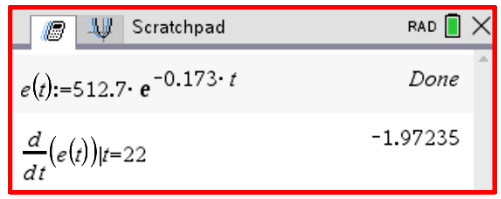
SOLUTIONS

Follow the directions to answer each of the following problems. Only use your calculator when a problem displays the calculator icon.

Topic 4.1 & 4.3 Interpreting the Meaning of a Derivative in Context; Rates of Change in Applied Contexts Other Than Motion

1. Eager rock fans enter a line to buy tickets to see the renowned band, Sir Isaac & the Newtones at a rate modeled by the function given by $E(t) = 512.7e^{-0.173t}$ where $E(t)$ is measured in people per minute and t is measured in minutes for the interval $0 \le t \le 30$. Find $E'(22)$ and using correct units, interpret its meaning in the context of the problem.

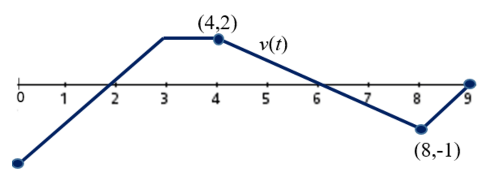
$E'(22) = -1.972$ people/minute / minute
The rate at which people are entering the line to buy tickets is decreasing by 1.972 people per minute per minute.



Topic 4.2 Straight Line Motion: Connecting Position, Velocity and Acceleration

2. The graph below shows the velocity, $v(t)$, of a particle moving along the x-axis and can be defined by a continuous linear piecewise-defined function over the interval $0 \le t \le 9$.

Note that $v'(t) = 0$ on $3 < x < 4$.



a. Over which time interval(s) does the particle move to the left? Justify your answer.

The particle moves to the left on the intervals $0 < t < 2$ and $6 < t < 9$ because $v(t) < 0$ on those intervals.

b. Over which time interval(s) is the particle speeding up? Justify your answer.

The particle is speeding up on the interval $2 < t < 3$ because $v(t) > 0$ and $a(t) > 0$ on that interval.
The particle is also speeding up on the interval $6 < t < 8$ because $v(t) < 0$ and $a(t) < 0$ on that interval.

c. Over which time interval(s) is the particle's speed decreasing? Justify your answer.

The particle's speed is decreasing on the interval $4 < t < 6$ because $v(t) > 0$ and $a(t) < 0$ on that interval.
The particle's speed is also decreasing on the interval $0 < t < 2$ and $8 < t < 9$ because $v(t) < 0$ and $a(t) > 0$ on that interval.

Topics 4.4 & 4.5: Related Rates

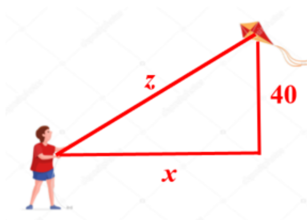
3. If $\sqrt{x}+y=6$ and $\frac{dy}{dt}=2$, find $\frac{dx}{dt}$ when $x=4$.

Given: $\frac{dy}{dt}=2$ Find: $\frac{dx}{dt}$ when $x=4$ Equation: $\sqrt{x}+y=6$

$\frac{1}{2}x^{-1/2}\cdot\frac{dx}{dt}+\frac{dy}{dt}=0 \quad\rightarrow\quad \frac{1}{2\sqrt{x}}\cdot\frac{dx}{dt}+\frac{dy}{dt}=0$

$\frac{1}{2\sqrt{4}}\cdot\frac{dx}{dt}+2=0 \quad\rightarrow\quad \frac{dx}{dt}=-2\cdot 2\sqrt{4}=-8$

4. A kite is flying at a height of 40 ft. A child is flying it so that it is moving horizontally at a rate of 3 ft/sec. If the string is taut, at what rate is the string being let out when the length of the string released is 50 ft?



Given: $\frac{dx}{dt}=3$ Find: $\frac{dz}{dt}$ when $z=50$ Equation: $x^2+40^2=z^2$

$2x\frac{dx}{dt}=2z\frac{dz}{dt}$

$2(30)(3)=2(50)\frac{dz}{dt}$

$\frac{dz}{dt}=\frac{180}{100}=\frac{9}{5}$ ft/sec

5. A spherical snowball is being made so that its volume is increasing at the rate of 8 cu. ft/min. Find the rate of change at which the radius is increasing when the snowball is 4 ft in diameter. The volume of a sphere is $V=\frac{4}{3}\pi r^3$.

Given: $\frac{dV}{dt}=8$ Find: $\frac{dr}{dt}$ when $r=2$ Equation: $V=\frac{4}{3}\pi r^3$

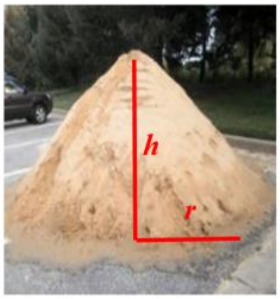
$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$

$8=4\pi(2)^2\frac{dr}{dt}$

$\frac{dr}{dt}=\frac{8}{16\pi}=\frac{1}{2\pi}$ ft/min

6. Sand is being dropped onto a conical pile at a rate of 10 cubic meters per minute. If the height of the pile always twice the base radius, at what rate is the height increasing when the pile is 8 m high?

The volume of a cone is $V = \frac{1}{3}\pi r^2 h.$



Given: $\frac{dV}{dt} = 10$ Find: $\frac{dh}{dt}$ when $h = 8$ Equation: $V = \frac{1}{3}\pi r^2 h$

$h = 2r$ $\qquad V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 \cdot h \Rightarrow V = \frac{\pi}{12}h^3$

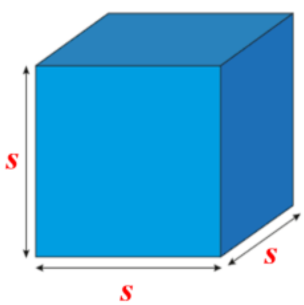
$r = \frac{h}{2}$

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

$$10 = \frac{\pi}{4}\cdot(8)^2\cdot\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{10\cdot 4}{64\pi} = \frac{5}{8\pi}\text{ m/min}$$

7. The volume of a cube is increasing by 10 cm^3/min. Find the rate the surface area is increasing when the side of the cube is 5 cm.



Given: $\frac{dh}{dt}$ Find: $\frac{dA}{dt}$ when $s = 5$ Equations: $V = s^3,\quad A = 6s^2$

First, look at the rate of change of the volume.

$$\frac{dV}{dt} = 3s^2\frac{ds}{dt}$$

$$10 = 3(5)^2\frac{ds}{dt}$$

$$\frac{ds}{dt} = \frac{10}{75} = \frac{2}{15}$$

Second, look at the rate of change of the surface area.

$$\frac{dA}{dt} = 12s\frac{ds}{dt}$$

$$\frac{dA}{dt} = 12(5)\left(\frac{2}{15}\right) = 8\text{ cm}^2\text{ / min}$$

8. Oil is leaking from a pipeline on the surface of a lake and forms an oil slick whose volume increases at a constant rate of 2000 cubic centimeters per minute. The oil slick takes the form of a right circular cylinder with both its radius and height changing with time. (Note: The volume V of a right circular cylinder with radius r and height h is given by $V = \pi r^2 h$.)

a.) At the instant when the radius of the oil slick is 100 centimeters and the height is 0.5 centimeter, the radius is increasing at the rate of 2.5 centimeters per minute. At this instant, what is the rate of change of the height of the oil slick with respect to time, in centimeters per minute?

Given: $\dfrac{dV}{dt} = 2000$ Find: $\dfrac{dh}{dt}$ when $r = 100,\ h = 0.5$ and $\dfrac{dr}{dt} = 2.5$ Equations: $V = \pi r^2 h$

$$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} \cdot h + \pi r^2 \cdot \frac{dh}{dt}$$

$$2000 = 2\pi(100)(2.5)(0.5) + \pi(100)^2 \frac{dh}{dt}$$

$$\pi(100)^2 \frac{dh}{dt} = 2000 - 2\pi(100)\left(\frac{5}{2}\right)\left(\frac{1}{2}\right)$$

$$\frac{dh}{dt} = \frac{2000 - 250\pi}{10{,}000\pi} = \frac{200 - 25\pi}{1000\pi} = \frac{8 - \pi}{40\pi} \text{ cm/min}$$

b.) A recovery device arrives on the scene and begins removing oil. The rate at which oil is removed is $R(t) = 400\sqrt{t}$ cubic centimeters per minute, where t is the time in minutes since the device began working. Oil continues to leak at a rate of 2000 cubic centimeters per minute. Find the time t when the oil slick is not changing volume.

Rate of change of the volume of the oil = leak rate - removal rate.

Need Leak Rate - Removal Rate = 0

$$2000 - 400\sqrt{t} = 0$$

$$2000 = 400\sqrt{t}$$

$$5 = \sqrt{t}$$

$t = 25$ minutes

Topic 4.6: Linear Approximation

9. Use the tangent line approximation for $f(x)=\sqrt{x}$ at $x=64$ to approximate $\sqrt{65}-\sqrt{63}$.

(A) 0 **(B)** $\frac{1}{32}$

(C) $\frac{1}{16}$ **(D)** $\frac{1}{8}$

$$f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}};\quad f'(64)=\frac{1}{2\sqrt{64}}=\frac{1}{16};\quad f(64)=\sqrt{64}=8$$

$$y-8=\frac{1}{16}(x-64)\ \rightarrow\ L(x)=8+\frac{1}{16}(x-64)$$

$$f(65)-f(63)\approx L(65)-L(63)=8+\frac{1}{16}(1)-\left(8+\frac{1}{16}(-1)\right)=\frac{1}{16}+\frac{1}{16}=\frac{2}{16}=\frac{1}{8}$$

10. Let g be a function given by $g(x)=x\cdot f(x)$. If $f(-1)=3$ and $f'(-1)=-2$, use the tangent line to g at $x=-1$ to approximate $g(-0.9)$.

(A) -2.5 **(B)** -0.2

(C) -1.8 **(D)** 3.5

$$g'(x)=1\cdot f(x)+x\cdot f'(x);\quad g'(-1)=f(-1)+(-1)\cdot f'(-1)=3-1(-2)=5$$

$$g(-1)=-1\cdot f(-1)=-1\cdot=-3$$

$$y-(-3)=5(x-(-1))\ \rightarrow\ L(x)=-3+5(x+1)$$

$$g(-0.9)\approx L(-0.9)=-3+5(-0.9+1)=-3+5\left(\frac{1}{10}\right)=-2.5$$

11. Given a function, $f(x)$, the linear approximation for $f(a+0.1)$ would be given by

(A) $f(a)+10f'(a)$ **(B)** $f(a)+\frac{f'(a)}{10}$

(C) $f'(a+0.1)-f'(a)$ **(D)** $10\left[f'(a+0.1)-f'(a)\right]$

$$L(x)=f(a)+f'(a)(x-a)$$

$$f(a+0.1)\approx L(a+0.1)=f(a)+f'(a)(a+0.1-a)$$

$$=f(a)+\frac{1}{10}f'(a)$$

12. Find the error using the linear approximation of $f(x)=(1-2x)^2$ at $x=1$ to approximate $f(0.9)$.

(A) 0.04 **(B)** 0.6

(C) 0.16 **(D)** 0.64

$f'(x)=2(1-2x)(-2)$ You must use the chain rule.

$$f'(1)=2(1-2(1))(-2)=4$$

$$f(1)=(1-2(1))^2=1$$

$$y-1=4(x-1)\ \rightarrow\ L(x)=1+4(x-1)$$

$$f(0.9)\approx L(0.9)=1+4(0.9-1)=1+4\left(-\frac{1}{10}\right)$$

$$=1-\frac{2}{5}=\frac{3}{5}=0.6$$

$$f(0.9)=(1-2(0.9))^2=0.64$$

Error $=|0.64-0.6|=0.04$

Topic 4.7: Indeterminate Forms & L'Hospital's Rule

13. What is $\lim\limits_{x\to 1} \dfrac{ex - e^x}{x^2 - 1}$?

(A) e **(B)** 1

(C) $\dfrac{e}{4}$ **(D)** 0

$\lim\limits_{x\to 1}\left(ex - e^x\right) = e - e = 0$

$\lim\limits_{x\to 1}\left(x^2 - 1\right) = 1 - 1 = 0$

L'Hospital's Rule can be applied.

$= \lim\limits_{x\to 1} \dfrac{e - e^x}{2x} = \dfrac{e - e}{2} = 0$

14. The function f is continuous and twice-differentiable for all values x, $f(0) = 1$, $f'(0) = 1$, and $f''(0) = 2$. What is the following limit? $\lim\limits_{x\to 0} \dfrac{f(x) - x - 1}{\sin(2x) - x^2 - 2x}$

(A) -1 **(B)** 0

(C) 1 **(D)** does not exist

Because $f(x)$ is differentiable, it is also continuous.

$\lim\limits_{x\to 0}\left(f(x) - x - 1\right) = 1 - 0 - 1 = 0$

$\lim\limits_{x\to 0}\left(\sin(2x) - x^2 - 2x\right) = 0 - 0 - 0 = 0$

L'Hospital's Rule can be applied.

$= \lim\limits_{x\to 0} \dfrac{f'(x) - 1}{2\cos(2x) - 2x - 2} = \dfrac{1 - 1}{2(1) - 0 - 2} = \dfrac{0}{0}$

L'Hospital's Rule can be applied again.

$= \lim\limits_{x\to 0} \dfrac{f''(x)}{-4\sin(2x) - 2} = \dfrac{2}{-4(0) - 2} = -\dfrac{2}{2} = -1$

15. Find the following limit: $\lim\limits_{t\to 0} \dfrac{e^{2t} - 1}{\sin t}$. Be sure to state any conditions that must be met.

$\lim\limits_{t\to 0}\left(e^{2t} - 1\right) = e^0 - 1 = 0$

$\lim\limits_{t\to 0}\left(\sin t\right) = 0$

L'Hospital's Rule can be applied.

$= \lim\limits_{t\to 0} \dfrac{2e^{2t}}{\cos t} = \dfrac{2e^0}{\cos(0)} = \dfrac{2}{1} = 2$

Note: In order to receive full credit for a Free Response Question, a student must show the first two steps seen to the left.