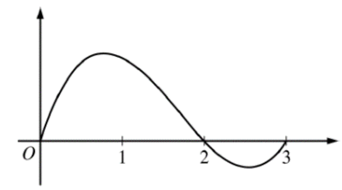
AP Calculus AB

UNIT 5 REVIEW – Analytical Applications of Differentiation.

You may use a graphing calculator for Questions 1-6.

1. The graph of the twice differentiable function f is shown in the figure. Which of the following correctly orders $f(2),\ f'(2),\ f''(2)$?



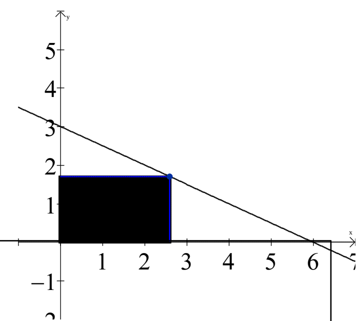
Graph of f

(A) $f(2) < f'(2) < f''(2)$ **(B)** $f''(2) < f(2) < f'(2)$

(C) $f'(2) < f(2) < f''(2)$ **(D)** $f(2) < f''(2) < f'(2)$

At $x = 2$, the graph of f is decreasing (first derivative negative), equal to zero (the function equals zero) and concave up (second derivative > 0)

2. A rectangle is bounded by the x- and y-axes and the graph of $y = \frac{6-x}{2}$ as in the figure below. What length and width should the rectangle have so that its area is a maximum?



(A) $x = 6,\ y = 2.5$ **(B)** $x = 3,\ y = 2.5$

(C) $x = 1.5,\ y = 3$ **(D)** $x = 3,\ y = 1.5$

Primary Equation: $A = xy$ Secondary Equation: $y = \frac{6-x}{2}$

$A = x\left(\frac{6-x}{2}\right) = 3x - \frac{1}{2}x^2$

$A' = 3 - x \quad \rightarrow \quad A' = 0 \text{ when } x = 3$ Because $A'' = -1 < 0$, the max area occurs when $x = 3$.

$y(3) = \frac{6-3}{2} = \frac{3}{2}$

3. Let $f(x)$ be a strictly increasing function such that $f(x) < 0$ for all values of x. Let $g(x)$ be a strictly decreasing function such that $g(x) > 0$ for all values of x. If $h(x) = f(x)g(x)$, which statement is true?

(A) $h(x) > 0$ and is strictly increasing. **(B)** $h(x) > 0$ and is strictly decreasing.

(C) $h(x) < 0$ and is strictly increasing. **(D)** $h(x) < 0$ and is strictly decreasing.

$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

If $f(x)$ is strictly increasing, then $f'(x) > 0$. If $g(x)$ is strictly decreasing, then $g'(x) < 0$.

Therefore, we have $h'(x) = (+)(+) + (-)(-) = \text{positive}$....and $h(x) = (+)(-)$.

Thus $h(x) < 0$ and is increasing.

4. A right circular cylinder can is to be designed to hold 22 cubic inches of a liquid. The cost for the material for the top and bottom of the can is \$0.02 per square inch while the cost for the material of the side of the can is \$0.01 per square inch. Let r represent the radius and h the height of the cylinder.

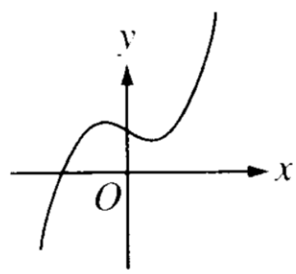
The lateral area (the sides) of a cylinder is $A = 2\pi rh$, the area of a circle is $A = \pi r^2$, and the volume of a cylinder is $V = pr^2h$. Which of the following represent the correct primary and secondary equations in order to minimize the total cost of manufacturing the can?

(A) P.E.: $C = .02(2\pi rh) + .01(\pi r^2)$
S.E.: $22 = \pi r^2 h$

(B) P.E.: $C = .01(2\pi rh) + .02(\pi r^2)$
S.E.: $22 = \pi r^2 h$

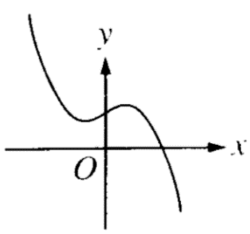
(C) P.E.: $C = .01(2\pi rh) + .02(2\pi r^2)$
S.E.: $22 = \pi r^2 h$

(D) P.E.: $C = .02(2\pi rh) + .01(2\pi r^2)$
S.E.: $22 = \pi r^2 h$

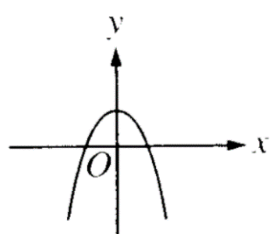


5. The graph of $y = h(x)$ is shown above. Which of the following could be the graph of $y = h'(x)$?

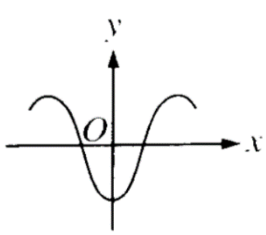
(A)



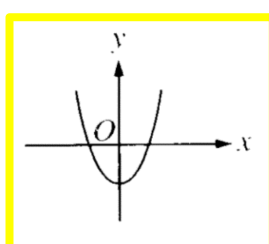
(B)



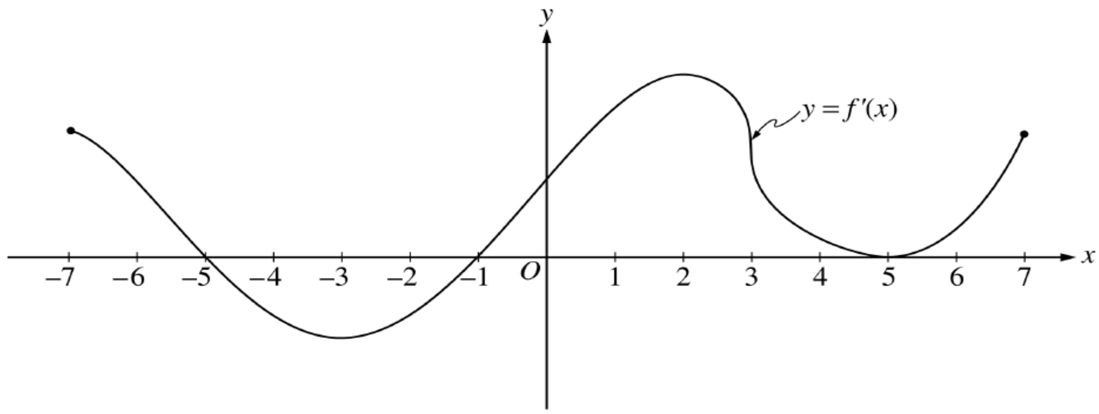
(C)



(D)



6. The figure below shows the graph of f', the derivative of the function *f*, for $-7 \le x \le 7$. The graph of f' has horizontal tangent lines at $x = -3$, $x = 2$, and , $x = 5$, and a vertical tangent at $x = 3$.



a. Find all values of *x*, for $-7 \le x \le 7$, at which *f* attains a relative minimum. Justify your answer.

The graph of $f(x)$ attains a relative minimum at $x = -1$ because $f'(-1) = 0$ and $f'(x)$ changes from negative to positive at $x = -1$.

b. Find all values of *x*, for $-7 \le x \le 7$, at which *f* attains a relative maximum. Justify your answer.

The graph of $f(x)$ attains a relative maximum at $x = -5$ because $f'(-5) = 0$ and $f'(x)$ changes from positive to negative at $x = -5$.

c. Find all intervals for which $f''(x) < 0$. Justify your answer.

$f''(x) < 0$ on the intervals $[-7,-3]$ and $[2,5]$ because $f'(x)$ is decreasing on those intervals. Note: you could have answered with open intervals: $(-7,-3)$ and $(2,5)$.

d. Find all values of *x* for $-7 \le x \le 7$ where $f(x)$ has a point of inflection. Justify your answer.

The graph of $f(x)$ has a point of inflection at $x = -3$, $x = 2$, and $x = 5$ because $f''(x) = 0$ at eah of those values and and $f''(x)$ changes signs at each of those values.

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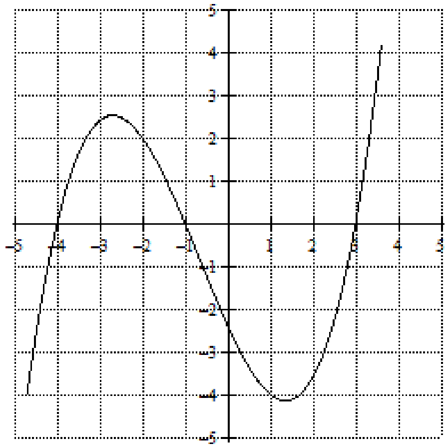
Do not use a calculator for Problems 7-17.

7. The graph of the **second derivative** of a function $f(x)$ is shown below. Which of the following is/are true?

I. The graph of $f(x)$ has a point of inflection at $x = -1$.

II. The graph of $f(x)$ is concave down on the interval $-1 < x < 3$.

III. The graph $f'(x)$ is increasing at $x = 2$.



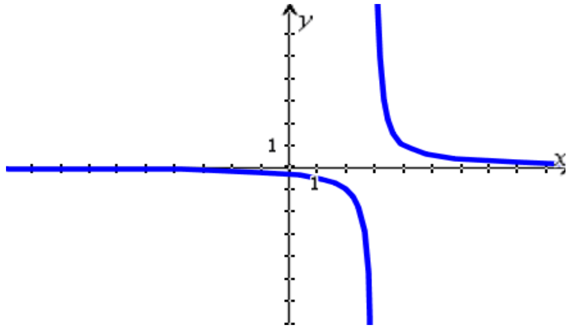
Graph of $f''(x)$

(A) I only **(B)** II only

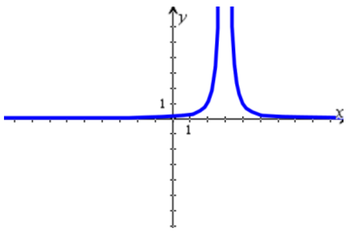
(C) I and II only **(D)** III only

There will be a point of inflection at $x = -1$ because $f''(-1) = 0$ and $f''(x)$ changes signs at $x = -1$.
The graph of $f(x)$ is concave down on $(-1, 3)$ because $f''(x) < 0$ on that interval.
The graph of $f'(x)$ cannot increase at $x = 1$ because $f''(x) < 0$ at $x = 1$.

8. Given the graph of the derivative, $f'(x)$, which of the following would correctly depict a possible graph of $f(x)$?



(A)



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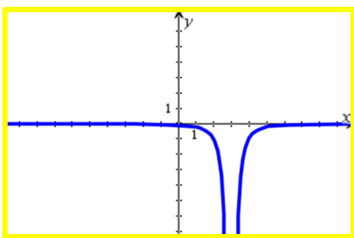
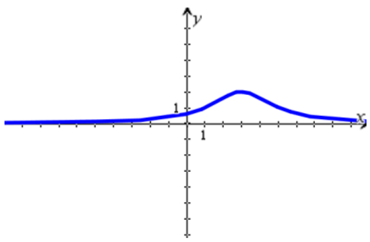
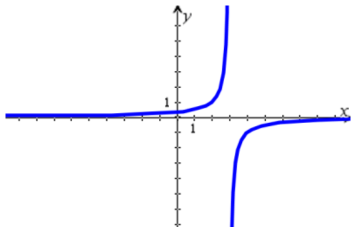


(C)



(D)



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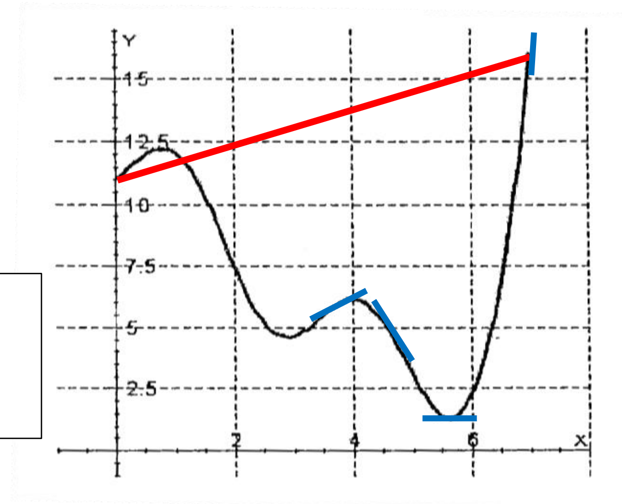
9. Use the graph of f on the right to estimate the value of c that justifies the Mean Value Theorem for the interval [0, 7].

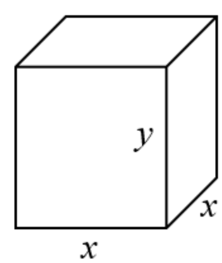


(A) 3.7 **(B)** 4.3

(C) 5.5 **(D)** 7

Of the three blue tangnet lines drawn, only the one at x = 3.7 looks as if could be parallel to the secant line drawn x = 0 to x = 7.

10. Engineers are designing a box-shaped aquarium with a square bottom and an open top. The aquarium must hold 500 ft³ of water. Which of the following represent the correct primary and secondary equations in order to find the dimensions they should use to create an aquarium with the least amount of glass?



(A) P.E.: $SA = x^2 y$ S.E.: $500 = 6xy$

(B) P.E.: $SA = 4xy + x^2$ S.E.: $500 = x^2 y$

(C) P.E.: $SA = 6xy$ S.E.: $500 = x^2 y$

(D) P.E.: $SA = x^2 y$ S.E.: $500 = 4xy + x^2$

11. Identify the open interval(s) on which the function $f(x) = x^2 - x - 12$ is increasing.

(A) $(-\infty, -3)$ and $(4, \infty)$ **(B)** $(-\infty, \infty)$

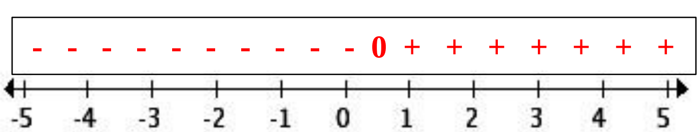
(C) $(-3, 4)$ **(D)** $\left(\frac{1}{2}, \infty\right)$

$f'(x) = 2x - 1$

$f'(x) = 0$

$2x - 1 = 0 \rightarrow x = \frac{1}{2}$

Sign of f'



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12. Suppose $f(x)$ is differentiable everywhere and $f(-2)=-5$ and $f'(x)\le 5$ for all values of x. Using the Mean Value Theorem, what is the largest possible value of $f(8)$?

(A) 35 **(B)** 45

(C) 55 **(D)** 65

The conclusion of the Mean Value Theorem says that $\dfrac{f(b)-f(a)}{b-a}=f'(c)$. .

Therefore $\dfrac{f(8)-f(-2)}{8-(-2)}$ must be less than or equal to 5.

$\dfrac{f(8)-f(-2)}{8-(-2)}\le 5 \rightarrow \dfrac{f(8)-(-5)}{10}\le 5 \rightarrow f(8)+5\le 50 \rightarrow f(8)\le 45$

13. For all x in the closed interval [2, 5], the function f has a positive first derivative and a negative second derivative. Which of the following could be a table of values for f?

(A)

x	$f(x)$
2	7
3	9
4	12
5	16

(B)

x	$f(x)$
2	7
3	11
4	14
5	16

(C)

x	$f(x)$
2	16
3	12
4	9
5	7

(D)

x	$f(x)$
2	16
3	14
4	11
5	7

A positive first derivative exists only in tables (A) and (B) becasue the values of $f(x)$ are increasing.
In order for the second derivative to be negative, the rate at which the $f(x)$ values increase must be decreasing (or slowing). This is evident in table (B).

14. On the closed interval $[0,2\pi]$, the absolute minimum of $f(x)=e^{\sin x}$ occurs at

(A) 0 **(B)** $\dfrac{\pi}{2}$

(C) $\dfrac{3\pi}{2}$ **(D)** 2π

$f'(x)=\cos x\cdot e^{\sin x}$

$\cos x\cdot e^{\sin x}=0$

$\cos x=0$ $\quad e^{\sin x}=0$

$x=\dfrac{\pi}{2},\dfrac{3\pi}{2}$ $\quad\varnothing$

x	$f(x)$
0	$e^{\sin(0)}=e^0=1$
$\dfrac{\pi}{2}$	$e^{\sin\left(\frac{\pi}{2}\right)}=e^1=e$
$\dfrac{3\pi}{2}$	$e^{\sin\left(\frac{3\pi}{2}\right)}=e^{-1}=\dfrac{1}{e}$
2π	$e^{\sin(2\pi)}=e^0=1$

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15. Let g be the function given by $g(x) = x^2 e^{kx}$, where k is a constant. For what value of k does g have a critical point at $x = \frac{2}{3}$?

(A) -3 **(B)** $-\frac{3}{2}$

(C) $-\frac{1}{3}$ **(D)** 0

$g'(x) = 2x \cdot e^{kx} + x^2 \cdot ke^{kx}$

Having a critical point at $x = \frac{2}{3}$ implies that $g'\left(\frac{2}{3}\right) = 0$

$$g'\left(\frac{2}{3}\right) = 2\left(\frac{2}{3}\right) \cdot e^{k\left(\frac{2}{3}\right)} + \left(\frac{2}{3}\right)^2 \cdot ke^{k\left(\frac{2}{3}\right)} = 0$$

$$\frac{4}{3}e^{\frac{2k}{3}} + \frac{4}{9}ke^{\frac{2k}{3}} = 0$$

$$4e^{\frac{2k}{3}}\left(\frac{1}{3} + \frac{1}{9}k\right) = 0$$

$$4e^{\frac{2k}{3}} = 0 \qquad \frac{1}{3} + \frac{1}{9}k = 0$$

$$\varnothing \qquad \frac{1}{9}k = -\frac{1}{3}$$

$$k = -3$$

16. The maximum value of $f(x) = 2x^3 - 15x^2 + 36x$ on the closed interval $[0, 4]$ is

(A) 28 **(B)** 30

(C) 32 **(D)** 48

$f'(x) = 6x^2 - 30x + 36$

$6x^2 - 30x + 36 = 0$

$6(x^2 - 5x + 6) = 0$

$6(x-3)(x-2) = 0$

$x = 2, 3$

x	$f(x)$
0	$2(0)^3 - 15(0)^2 + 36(0) = 0$
2	$2(2)^3 - 15(2)^2 + 36(2) = 16 - 60 + 72 = 28$
3	$2(3)^3 - 15(3)^2 + 36(3) = 54 - 135 + 108 = 27$
4	$2(4)^3 - 15(4)^2 + 36(4) = 128 - 240 + 144 = 32$

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17. Consider the function f defined by $f(x)=\dfrac{x^2}{x^2+3}$ whose first and second derivatives are given by

$f'(x)=\dfrac{6x}{(x^2+3)^2}$ and $f''(x)=\dfrac{18(1-x^2)}{(x^2+3)^3}$ respectively.

a. Use $f'(x)$ to find any critical numbers.

$6x=0$
$x=0$

b. Construct a chart or number line and use the First Derivative Test to identify (state) the intervals of increasing and decreasing and state the location of any extrema.

Interval	$(-\infty,0)$	$(0,\infty)$
Test	$f'(-1)=\frac{-6}{16}$	$f'(1)=\frac{6}{16}$
Sign	negative	positive
Conclusion	decreasing	increasing

$f(x)$ is decreasing on $(-\infty,0)$ because $f'(x)<0$
$f(x)$ is increasing on $(0,\infty)$ because $f'(x)>0$
There is a relative minimum at $x = 0$ b/c $f'(x)$ changes from negative to positive at $x = 0$.

c. Solve $f''(x)=0$.

$1-x^2=0$
$x=\pm 1$

d. Construct a chart or number line and use the Test for Concavity to determine (state) the intervals of concavity and state any point(s) of inflection.

Interval	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
Test	$f''(-2)=\frac{-18(3)}{7^3}$	$f''(0)=\frac{-18(-1)}{3^3}$	$f''(2)=\frac{-18(3)}{7^3}$
Sign	negative	positive	negative
Conclusion	concave down	concave up	concave down

$f(x)$ is concave down on $(-\infty,-1)$ and $(1,\infty)$ because $f''(x)<0$ on those intervals.
$f(x)$ is concave up on $(-1,1)$ because $f''(x)>0$ on that interval.
There are poi's at x = -1 and 1 $f'(x)$ changes signs at those values.

e. Construct the graph of f on the coordinate plane to the right.