Skill Builder: Topic 6.10 – Integration Using Long Division and Completing the Square

Find the integral of each of the following.

1. $\displaystyle\int \frac{x^2-3x+2}{x+1}\,dx$

$$\begin{array}{r} x-4 \\ x+1\overline{)x^2-3x+2} \\ \underline{-(x^2+x)} \\ -4x+2 \\ \underline{-(-4x-4)} \\ 6 \end{array}$$

$$\int \frac{x^2-3x+2}{x+1}\,dx = \int\left(x-4+\frac{6}{x+1}\right)dx$$

$$=\frac{1}{2}x^2-4x+6\ln|x+1|+C$$

2. $\displaystyle\int \frac{x^2+4x}{x^2+4x+13}\,dx$

$$\begin{array}{r} 1 \\ x^2+4x+13\overline{)x^2+4x} \\ \underline{-(x^2+4x+13)} \\ -13 \end{array}$$

$$\int \frac{x^2+4x}{x^2+4x+13}\,dx = \int\left(1-\frac{13}{x^2+4x+13}\right)dx$$

$$=\int\left(1-\frac{13}{x^2+4x+4+13-4}\right)dx$$

$$=\int\left(1-\frac{13}{(x+2)^2+9}\right)dx$$

$$=x-\frac{13}{3}\arctan\left(\frac{x+2}{3}\right)+C$$

3. $\displaystyle\int \frac{x^3-3x^2+5}{x-3}\,dx$

$$\begin{array}{r} x^2 \\ x-3\overline{)x^3-3x^2+5} \\ \underline{-(x^3-3x^2)} \\ 5 \end{array}$$

$$\int \frac{x^3-3x^2+5}{x-3}\,dx = \int\left(x^2+\frac{5}{x-3}\right)dx$$

$$=\frac{x^3}{3}+5\ln|x-3|+C$$

4. $\displaystyle\int \frac{x^4+x-4}{x^2+2}\,dx$

$$\begin{array}{r} x^2-2 \\ x^2+2\overline{)x^4+x-4} \\ \underline{-(x^4+2x^2)} \\ -2x^2+x-4 \\ \underline{-(-2x^2-4)} \\ x \end{array}$$

$$\int \frac{x^4+x-4}{x^2+2}\,dx = \int\left(x^2-2+\frac{x}{x^2+2}\right)dx$$

$$=\frac{x^3}{3}-2x+\frac{1}{2}\ln\left(x^2+2\right)+C$$

5. $\displaystyle\int \frac{dx}{4+(x-1)^2}$

$a=2 \qquad u=x-1$

$\qquad\qquad du=dx$

$\displaystyle\int \frac{dx}{4+(x-1)^2}$ results in an "arctan" form

$$\int \frac{dx}{4+(x-1)^2} = \frac{1}{2}\arctan\left(\frac{x-1}{2}\right)+C$$

6. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$

$a=2 \qquad u=x$

$\qquad\qquad du=dx$

$\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$ results in an "arcsin" form

$$\int \frac{dx}{\sqrt{4-x^2}} = \arcsin\left(\frac{x}{2}\right)+C$$

7. $\displaystyle\int \frac{dx}{x^2-2x+2}$

$$\int \frac{dx}{x^2-2x+1+2-1} = \int \frac{dx}{(x-1)^2+1}$$

$u=x-1 \qquad a=1$

$du=dx$

$\displaystyle\int \frac{dx}{(x-1)^2+1}$ results in an "arctan" form

$$\int \frac{dx}{(x-1)^2+1} = \arctan(x-1)+C$$

8. $\displaystyle\int \frac{2x}{x^2+6x+13}\,dx$

One way to solve this problem is to notice what the derivative of the denominator would produce. Consequently, we would hope the numerator consisted of $2x+6$. We can make that happen by adding 6 and then immediately subtratcting 6.

$$\begin{aligned}
\int \frac{2x}{x^2+6x+13}\,dx &= \int \frac{2x+6-6}{x^2+6x+13}\,dx \\
&= \int \frac{2x+6}{x^2+6x+13}\,dx - \int \frac{6}{x^2+6x+13}\,dx \\
&= \int \frac{2x+6}{x^2+6x+13}\,dx - \int \frac{6}{x^2+6x+9+13-9}\,dx \\
&= \int \frac{2x+6}{x^2+6x+13}\,dx - \int \frac{6}{(x+3)^2+4}\,dx \\
&= \ln\left|x^2+6x+13\right| - \frac{6}{2}\arctan\left(\frac{x+3}{2}\right)+C \\
&= \ln\left|x^2+6x+13\right| - 3\arctan\left(\frac{x+3}{2}\right)+C
\end{aligned}$$

9. $\displaystyle\int \frac{3}{\sqrt{4x-x^2}}\,dx$

$$\begin{aligned}
\int \frac{3}{\sqrt{4x-x^2}}\,dx &= \int \frac{3}{\sqrt{-(x^2-4x+4)+4}}\,dx \\
&= \int \frac{3}{\sqrt{4-(x-2)^2}}\,dx \\
&= 3\arcsin\left(\frac{x-2}{2}\right)+C
\end{aligned}$$

10. $\displaystyle\int \frac{1}{\sqrt{-x^2-x}}\,dx$

$$\begin{aligned}
\int \frac{1}{\sqrt{-x^2-x}}\,dx &= \int \frac{1}{\sqrt{-\left(x^2+x+\frac{1}{4}\right)+\frac{1}{4}}}\,dx \\
&= \int \frac{1}{\sqrt{\frac{1}{4}-\left(x+\frac{1}{2}\right)^2}}\,dx \\
&= \arcsin\left(\frac{x+\frac{1}{2}}{\frac{1}{2}}\right)+C \text{ or } \arcsin(2x+1)+C
\end{aligned}$$