Skill Builder: Topic 6.3A – Finding Areas Using the Limit Process

Find the limit of $s(n)$ as n $\rightarrow\infty$. Try without performing any simplification.

1.) $s(n)=\frac{81}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]$

$$\lim_{n\to\infty}\left(\frac{81}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]\right)=\frac{81}{4}$$

2.) $s(n)=\frac{64}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$

$$\lim_{n\to\infty}\left(\frac{64}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\right)=\frac{128}{6}=\frac{64}{3}$$

3.) $s(n)=\frac{18}{n^2}\left[\frac{n(n+1)}{2}\right]$

$$\lim_{n\to\infty}\left(\frac{18}{n^2}\left[\frac{n(n+1)}{2}\right]\right)=\frac{18}{2}=9$$

4.) $s(n)=\frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]$

$$\lim_{n\to\infty}\left(\frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]\right)=\frac{1}{2}$$

Find the formula for the sum of *n* terms. Use the formula to find the limit as n →∞.

5.) $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$

$$=\lim_{n\to\infty}\left[\frac{4}{n^2}\cdot\sum_{i=1}^{n} i\right]$$

$$=\lim_{n\to\infty}\left(\frac{4}{n^2}\cdot\frac{n(n+1)}{2}\right)$$

$$=\lim_{n\to\infty}\left(\frac{4n^2+4n}{2n^2}\right)$$

$$=\frac{4}{2}=2$$

6.) $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)^3\left(\frac{2}{n}\right)$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{n+2i}{n}\right)^3\left(\frac{2}{n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{2}{n^4}\cdot\sum_{i=1}^{n}\left(n^3+3n^2(2i)^1+3n(2i)^2+(2i)^3\right)\right)$$

$$=\lim_{n\to\infty}\left(\frac{2}{n^4}\cdot\sum_{i=1}^{n}\left(n^3+6n^2i+12ni^2+8i^3\right)\right)$$

$$=\lim_{n\to\infty}\frac{2}{n^4}\left[n^3\cdot n+6n^2\cdot\frac{n(n+1)}{2}+12n\cdot\frac{n(n+1)(2n+1)}{6}+8\cdot\frac{n^2(n+1)^2}{4}\right]$$

$$=\lim_{n\to\infty}\frac{2}{n^4}\left[n^4+\frac{6n^3(n+1)}{2}+\cdot\frac{12n^2(n+1)(2n+1)}{6}+\frac{8n^2(n+1)^2}{4}\right]$$

$$=\lim_{n\to\infty}\frac{2}{n^4}\left[\frac{12\cdot n^4}{12}+\frac{6\cdot 6n^3(n+1)}{6\cdot 2}+\cdot\frac{2\cdot 12n^2(n+1)(2n+1)}{2\cdot 6}+\frac{3\cdot 8n^2(n+1)^2}{3\cdot 4}\right]$$

$$=\lim_{n\to\infty}\frac{2}{n^4}\left[\frac{12n^4+36n^4+36n^3+24n^2(2n^2+3n+1)+24n^2(n^2+2n+1)}{12}\right]$$

$$=\lim_{n\to\infty}\frac{2}{n^4}\left[\frac{12n^4+36n^4+36n^3+48n^4+72n^3+24n^2+24n^4+48n^3+24n^2}{12}\right]$$

$$=\lim_{n\to\infty}\frac{2}{n^4}\left[\frac{120n^4+156n^3+48n^2}{12}\right]=\frac{240}{12}=20$$

Use the limit process to find the area of the region between the graph of the function and the x-axis over the given interval.

7.) $f(x)=4x-2, \quad [1,4]$

$$\text{width} = \frac{4-1}{n}=\frac{3}{n}$$

$$\text{length} = f\left(1+\frac{3}{n}i\right)=4\left(1+\frac{3}{n}i\right)-2=4\left(\frac{n+3i}{n}\right)-2$$

$$=\frac{4n+12i}{n}-2\cdot\frac{n}{n}=\frac{2n+12i}{n}$$

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{3}{n}\right)\left(\frac{2n+12i}{n}\right)\right]$$

$$=\lim_{n\to\infty}\left(\frac{3}{n^2}\right)\sum_{i=1}^{n}(2n+12i)$$

$$=\lim_{n\to\infty}\left[\left(\frac{3}{n^2}\right)\left(2n\cdot n+12\cdot\frac{n(n+1)}{2}\right)\right]$$

$$=\lim_{n\to\infty}\left[\left(\frac{3}{n^2}\right)\left(2n^2+6n(n+1)\right)\right]$$

$$=\lim_{n\to\infty}\left[\left(\frac{3}{n^2}\right)\left(2n^2+6n^2+6n\right)\right]$$

$$=\lim_{n\to\infty}\left[\left(\frac{3}{n^2}\right)\left(8n^2+6n\right)\right]$$

$$=\lim_{n\to\infty}\left(\frac{24n^2+18n}{n^2}\right)$$

$$=24$$

8.) $f(x)=x^2+x, \quad [2,4]$

$$\text{width} = \frac{4-2}{n}=\frac{2}{n}$$

$$\text{length} = f\left(2+\frac{2}{n}i\right)=\left(2+\frac{2}{n}i\right)^2+\left(2+\frac{2}{n}i\right)$$

$$=\left(\frac{2n+2i}{n}\right)^2+\left(\frac{2n+2i}{n}\right)=\frac{4n^2+8ni+4i^2}{n^2}+\frac{2n+2i}{n}$$

$$=\frac{4n^2+8ni+4i^2}{n^2}+\frac{2n+2i}{n}\cdot\frac{n}{n}=\frac{4n^2+8ni+4i^2}{n^2}+\frac{2n^2+2ni}{n^2}$$

$$=\frac{6n^2+10ni+4i^2}{n^2}$$

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{2}{n}\right)\left(\frac{6n^2+10ni+4i^2}{n^2}\right)\right]$$

$$=\lim_{n\to\infty}\left(\frac{2}{n^3}\right)\sum_{i=1}^{n}\left(6n^2+10ni+4i^2\right)$$

$$=\lim_{n\to\infty}\left(\frac{2}{n^3}\right)\left(6n^2\cdot n+10n\cdot\frac{n(n+1)}{2}+4\cdot\frac{n(n+1)(2n+1)}{6}\right)$$

$$=\lim_{n\to\infty}\left(\frac{2}{n^3}\right)\left(6n^3+5n^2(n+1)+\frac{2n(n+1)(2n+1)}{3}\right)$$

$$=\lim_{n\to\infty}\left(\frac{2}{n^3}\right)\left(6n^3+5n^3+5n^2+\frac{4n^3+6n^2+2n}{3}\right)$$

$$=\lim_{n\to\infty}\left(\frac{2}{n^3}\right)\left(\frac{18n^3+15n^3+15n^2+4n^3+6n^2+2n}{3}\right)$$

$$=\lim_{n\to\infty}\left(\frac{2}{n^3}\right)\left(\frac{37n^3+21n^2+2n}{3}\right)$$

$$=\lim_{n\to\infty}\left(\frac{74n^3+42n^2+5n}{3n^3}\right)$$

$$=\frac{74}{3}$$

9.) $f(x)=2x^2-3x+1, \quad [1,2]$

$$\text{width} = \frac{2-1}{n}=\frac{1}{n}$$

$$\text{length} = f\left(1+\frac{1}{n}i\right)=2\left(1+\frac{1}{n}i\right)^2-3\left(1+\frac{1}{n}i\right)+1$$

$$=2\left(\frac{n+i}{n}\right)^2-3\left(\frac{n+i}{n}\right)+1=2\left(\frac{n^2+2ni+i^2}{n^2}\right)-3\left(\frac{n+i}{n}\right)+1$$

$$=2\left(\frac{n^2+2ni+i^2}{n^2}\right)-3\left(\frac{n+i}{n}\right)\cdot\frac{n}{n}+1\cdot\frac{n^2}{n^2}$$

$$=\frac{2n^2+4ni+2i^2}{n^2}-\frac{3n^2+3ni}{n^2}+\frac{n^2}{n^2}$$

$$=\frac{ni+2i^2}{n^2}$$

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{1}{n}\right)\left(\frac{ni+2i^2}{n^2}\right)\right]$$

$$=\lim_{n\to\infty}\left(\frac{1}{n^3}\right)\sum_{i=1}^{n}\left(ni+2i^2\right)$$

$$=\lim_{n\to\infty}\left(\frac{1}{n^3}\right)\left(n\cdot\frac{n(n+1)}{2}+2\cdot\frac{n(n+1)(2n+1)}{6}\right)$$

$$=\lim_{n\to\infty}\left(\frac{1}{n^3}\right)\left(\frac{n^3+n^2}{2}\cdot\frac{3}{3}+\frac{4n^3+6n^2+2n}{6}\right)$$

$$=\lim_{n\to\infty}\left(\frac{1}{n^3}\right)\left(\frac{3n^3+3n^2+4n^3+6n^2+2n}{6}\right)$$

$$=\lim_{n\to\infty}\left(\frac{1}{n^3}\right)\left(\frac{7n^3+9n^2+2n}{6}\right)$$

$$=\lim_{n\to\infty}\left(\frac{7n^3+9n^2+2n}{6n^3}\right)$$

$$=\frac{7}{6}$$

10.) $f(x)=9-x^2,$ between the x-intercepts

Between the x-intercepts means that we will find the area between x = -3 and x = 3.
However, due to the symmetrical nature of the graph, it is easier to find the area between
x = 0 and x = 3 and double the result.

$$\text{width} = \frac{3-0}{n}=\frac{3}{n}$$

$$\text{length} = f\left(\frac{3}{n}i\right)=9-\left(\frac{3}{n}i\right)^2=9-\frac{9i^2}{n^2}$$

$$=\frac{9n^2-9i^2}{n^2}$$

$$2\cdot\text{Area} = 2\cdot\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{3}{n}\right)\left(\frac{9n^2-9i^2}{n^2}\right)\right]$$

$$=2\cdot\lim_{n\to\infty}\left(\frac{27}{n^3}\right)\sum_{i=1}^{n}\left(n^2-i^2\right)$$

$$=2\cdot\lim_{n\to\infty}\left(\frac{27}{n^3}\right)\left(n^2\cdot n-\frac{n(n+1)(2n+1)}{6}\right)$$

$$=2\cdot\lim_{n\to\infty}\left(\frac{27}{n^3}\right)\left(n^3-\frac{(2n^3+3n^2+n)}{6}\right)$$

$$=2\cdot\lim_{n\to\infty}\left(\frac{27}{n^3}\right)\left(\frac{6n^3-2n^3-3n^2-n}{6}\right)$$

$$=2\cdot\lim_{n\to\infty}\left(\frac{27}{n^3}\right)\left(\frac{4n^3-3n^2-n}{6}\right)$$

$$=2\cdot\lim_{n\to\infty}\left(\frac{9}{n^3}\right)\left(\frac{4n^3-3n^2-n}{2}\right)$$

$$=2\cdot\lim_{n\to\infty}\left(\frac{36n^3-27n^2-9n}{2n^3}\right)$$

$$=2\cdot 18$$

$$=36$$

11.) $f(x)=x^3, \quad [0,4]$

$$\text{width} = \frac{4-0}{n} = \frac{4}{n}$$

$$\text{length} = f\left(\frac{4}{n}i\right) = \left(\frac{4}{n}i\right)^3 = \frac{64i^3}{n^3}$$

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{4}{n}\right)\left(\frac{64i^3}{n^3}\right)\right]$$

$$= \lim_{n\to\infty}\left(\frac{256}{n^4}\right)\sum_{i=1}^{n}\left(i^3\right)$$

$$= \lim_{n\to\infty}\left[\left(\frac{256}{n^4}\right)\left(\frac{n^2(n+1)^2}{4}\right)\right]$$

$$= \lim_{n\to\infty}\left[\left(\frac{256}{n^4}\right)\left(\frac{n^4+2n^3+n^2}{4}\right)\right]$$

$$= \lim_{n\to\infty}\left(\frac{256n^4+512n^3+256n^2}{4n^4}\right)$$

$$= \frac{256}{4}$$

$$= 64$$

It may be important to note that it is not necessary that students completely simplify the expression for area prior to taking its limit. Students may take the limit of the expression at any point they feel comfortable. Experiment by taking the limit at various stages and check your answer.