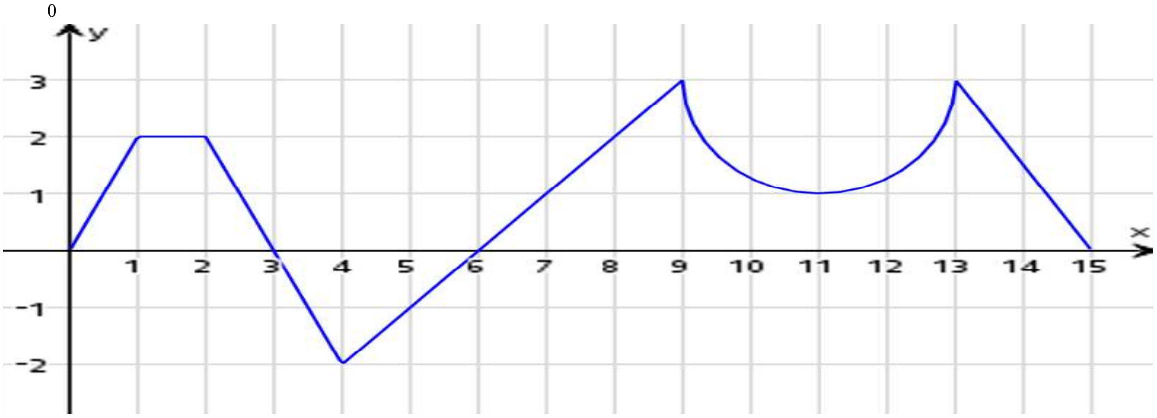
Skill Builder: Topics 6.4 & 6.5 – The Fundamental Theorem of Calculus and Accumulation Functions Interpreting the Behavior of Accumulation Functions Involving Area

1. Let f be a function whose graph consists of 5 line segments and a semicircle as shown in the figure below.

   Let $g(x)=\int_0^x f(t)\,dt$.



Find each of the following values.

a. $g(0)$

$=\int_0^0 f(t)\,dt=0$

b. $g(2)$

$=\int_0^2 f(t)\,dt=\frac{1}{2}(2+1)(2)=3$

c. $g(4)$

$=\int_0^4 f(t)\,dt=3+\int_2^4 f(t)\,dt$

$=3+\frac{1}{2}(2)(1)-\frac{1}{2}(2)(1)=3$

d. $g(6)$

$=\int_0^6 f(t)\,dt=3+\int_4^6 f(t)\,dt$

$=3-\frac{1}{2}(2)(2)=3-2=1$

e. $g(11)$

$=\int_0^6 f(t)\,dt=1+\int_6^{11} f(t)\,dt$

$=1+\frac{1}{2}(3)(3)+(2)(3)-\frac{1}{4}\pi(2)^2$

$=7+\frac{9}{2}-\pi=\frac{23-2\pi}{2}$

f. $g(15)$

$=\int_0^{11} f(t)\,dt=\frac{23-2\pi}{2}+\int_{11}^{15} f(t)\,dt$

$=\frac{23-2\pi}{2}+(2)(3)-\frac{1}{4}\pi(2)^2+\frac{1}{2}(2)(3)$

$=\frac{23-2\pi}{2}+9-\pi=\frac{41-4\pi}{2}$

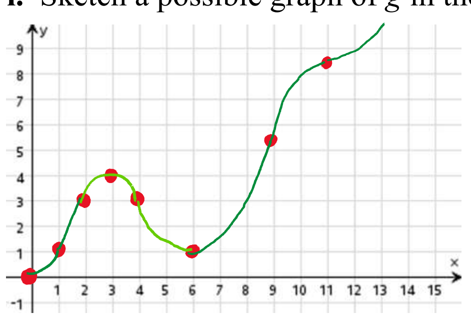
g. $g'(3)$

$=f(3)=0$

h. $g'(13)$

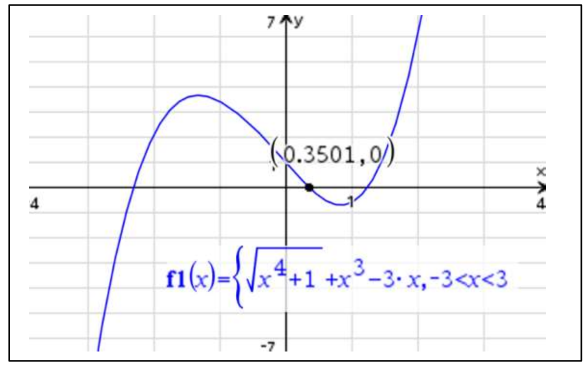
$=f(13)=3$

i. Sketch a possible graph of g in the coordinate plane below.



2. If $G(x)=\int_{2}^{x}\left(\sqrt{t^4+1}+t^3-3t\right)dt$, find a value x, $-3\le x\le 3$, for which G has a local maximum. Explain your reasoning.

$G(x)$ has a relative (local) maximum when $G'(x)=\sqrt{x^4+1}+x^3-3x$ changes from a positive to a negative. This occurs at $x=0.350$.



3. For $0\le t\le 18$, oil enters a tank at the rate of $E(t)$ gallons per hour, and at the same time oil leaves the tank at the rate of $L(t)$ gallons per hour, where t is measured in hours past midnight. At midnight, there are 2000 gallons of oil in the tank. Using each component of O, explain what the function $O(t)=2000+\int_{0}^{t}\left(E(x)-L(x)\right)dx$ represents in the context of the problem.

$O(t)$ represents the total amount of oil in the tank at time t.

4. If $F'(x)=f(x)$ and if $F(2)=3$, $F(1)=-1$, and $F(0)=2$, then $\int_{0}^{1}f(x)\,dx-\int_{1}^{2}f(x)\,dx$ equals which of the following?

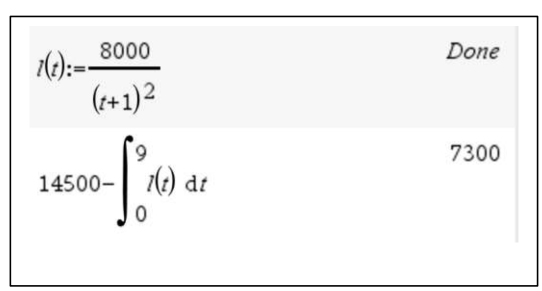
(A) 3 **(B)** -3 **(C)** -5 **(D)** -7

$$\int_{0}^{1}f(x)\,dx-\int_{1}^{2}f(x)\,dx=F(1)-F(0)-\left(F(2)-F(1)\right)$$
$$=(-1)-2-(3-(-1))$$
$$=-3-4=-7$$

5. Fans are leaving Gainbridge Fieldhouse after a big In diana Pacers win at a rate modeled by the function $L(t)=\dfrac{8000}{(t+1)^2}$, where L is the number of people leaving per minute t minutes after the conclusion of the game. If 14,500 people were present at the conclusion of the game, how many people were still in the arena 9 minutes after the conclusion of the game?

Let $P(t)$ represent the number of people on the arena at time t minutes after the concluson of the game.

$P(9)=14,500-\int_{0}^{9}L(t)\,dt=7300$ people



6. The volume of a balloon expands as the air inside the balloon is heated. The radius of the balloon, in centimeters, is modeled by the twice-differentiable functions $r(t)$, where t is measured in minutes. The table below gives selected values of the rate of change, $r'(t)$, of the radius of the balloon for $0 \le t \le 30$.

t	0	5	10	15	20	25	30
$r'(t)$	6.7	5.6	4.9	4.0	3.4	2.7	2.2

Use a midpoint Riemann sum with three subintervals to approximate $\int_0^{30} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{30} r'(t)\,dt$ in the context of this problem.

$$\int_0^{30} r'(t)\,dt \approx (10)(5.6)+(10)(4.0)+(10)(2.7)$$

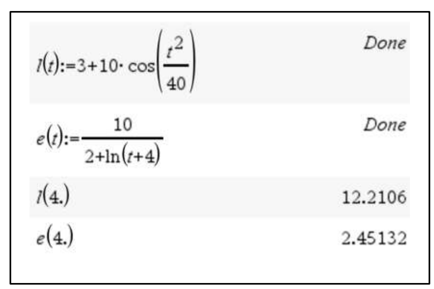
$$\approx 56+40+27$$

$$\approx 123$$

The balloon's radius grew by a total of 123 centimeters during the 30 minutes.

7. A storage tank contains 225 gallons of solvent at time $t=0$. During the interval $0 \le t \le 10$, solvent is being removed from the tank at a rate of $L(t) = 3 + 10\cos\left(\frac{t^2}{40}\right)$ gallons per hour. During this same interval, clean solvent is being pumped into the tank at a rate of $E(t) = \frac{10}{2+\ln(t+4)}$ gallons per hour.

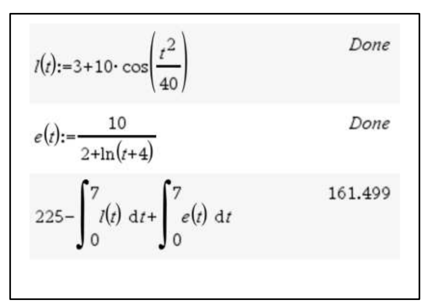
a. Is the amount of solvent in the tank increasing or decreasing a time $t = 4$ hours? Explain your reasoning.



$L(4) \approx 12.2106$ and $E(4) \approx 2.4513$

Since $L(4) > E(4)$, the amount of solvent in the tank is decreasing at time $t = 4$ hours.

c. How many gallons of solvent are in the tank at time $t = 7$ hours?



Let $S(t)$ represent the amount of solvent in the tank at time t.

$$S(t) = 225 - \int_0^t 3+10\cos\left(\frac{x^2}{40}\right)dx + \int_0^t \frac{10}{2+\ln(x+4)}dx$$

$$S(7) = 225 - \int_0^7 3+10\cos\left(\frac{x^2}{40}\right)dx + \int_0^7 \frac{10}{2+\ln(x+4)}dx$$

$$\approx 161.499 \text{ gallons}$$

8. On a certain day, the depth of snow at Paoli Peaks Ski Resort melts at a rate modeled by the function $M(t)$ given by $M(t) = 3\pi \sin\left(\frac{\pi t}{12}\right)$.

A snowmaking machine adds snow at a rate modeled by the function $S(t)$ given by $S(t) = 0.14t^3 - 0.16t^2 + 0.54t - 0.1$.

Both M and S are measured in inches per hour and t is measured in hours for $0 \le t \le 5$. At time $t = 0$, the snow depth is 32 inches.

a. How many inches of snow will melt during the 5-hour period?

$$\int_0^5 M(t)\,dt \approx 26.683 \text{ inches}$$

$m(t):=3\cdot\pi\cdot\sin\left(\frac{\pi\cdot t}{12}\right)$	*Done*
$s(t):=0.14\cdot t^3-0.16\cdot t^2+0.54\cdot t-0.1$	*Done*
$\int_0^{5.} m(t)\,dt$	26.6825

b. Express $A(t)$, the total depth of snow, in inches, at time t, as a function defined by an integral.

$$A(t) = 32 - \int_0^t M(x)\,dx + \int_0^t S(x)\,dx$$

c. Find the rate of change of the snow depth at time $t = 5$.

$A'(t) = -M(t) + S(t)$ or $S(t) - M(t)$

$A'(5) = S(5) - M(5)$

≈ 6.996 inches / hour

$m(t):=3\cdot\pi\cdot\sin\left(\frac{\pi\cdot t}{12}\right)$	*Done*
$s(t):=0.14\cdot t^3-0.16\cdot t^2+0.54\cdot t-0.1$	*Done*
$s(5.)-m(5.)$	6.99636

d. For $0 \le t \le 5$, at what time t is the depth of the snow a minimum? What is the minimum value? Justify your answer.

$A'(t) = 0$ when $t \approx 3.935$

Candidates: 0, 3.935, 5

t	$A(t)$
0	$A(0) = 32 - \int_0^0 M(x)\,dx + \int_0^0 S(x)\,dx = 32$
3.935	$A(3.935) = 32 - \int_0^{3.935} M(x)\,dx + \int_0^{3.935} S(x)\,dx \approx 23.457$
5	$A(5) = 32 - \int_0^5 M(x)\,dx + \int_0^5 S(x)\,dx \approx 26.776$

The snow reaches a minimum depth of approx. 23.457 inches at time t = 3.935 hours.

$m(t):=3\cdot\pi\cdot\sin\left(\frac{\pi\cdot t}{12}\right)$	*Done*
$s(t):=0.14\cdot t^3-0.16\cdot t^2+0.54\cdot t-0.1$	*Done*
solve$(s(t)-m(t)=0.,t)$	
$t=-2.91899$ or $t=-0.052121$ or $t=3.93567$	

$32-\int_0^{3.935} m(t)\,dt+\int_0^{3.935} s(t)\,dt$	23.4572
$32-\int_0^{5} m(t)\,dt+\int_0^{5} s(t)\,dt$	26.7758