UNIT 6 (Topics 6.7 - 6.10) EXAM REVIEW – II

Logarithmic, Exponential and Inverse Trigonometric Functions and Integration

Find each indefinite integral.

1. $\int \frac{1}{x\ln(x^2)}dx$

$$u = \ln\left(x^2\right)$$

$$du = \frac{1}{x^2}\left(2x\right)dx$$

$$du = \frac{2}{x}dx$$

$$\frac{x}{2}du = dx$$

$$\int \frac{1}{x\ln\left(x^2\right)}dx$$

$$= \int \frac{1}{xu}\frac{x}{2}du = \frac{1}{2}\int \frac{1}{u}du$$

$$= \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln\left|\ln\left(x^2\right)\right| + C}$$

2. $\int \frac{1}{x^{2/3}(1+\sqrt[3]{x})}dx$

$$\int \frac{1}{x^{2/3}\left(1+\sqrt[3]{x}\right)}dx$$

$$u = 1 + x^{1/3}$$

$$du = \tfrac{1}{3}x^{-2/3}dx$$

$$3x^{2/3}du = dx$$

$$= \int \frac{1}{x^{2/3}u}3x^{2/3}du = 3\int \frac{1}{u}du$$

$$= 3\ln|u| + C = \boxed{3\ln\left|1+\sqrt[3]{x}\right| + C}$$

3. $\int \csc 5\theta \;\; d\theta$

$$= \frac{1}{5}\int \csc\underbrace{\left(5\theta\right)}_{u}\underbrace{\left(5d\theta\right)}_{du} = \frac{1}{5}\int \csc u\,du$$

$$= \underbrace{\frac{1}{5}\left(-\ln\left|\csc u + \cot u\right|\right)}_{\text{Table of Integrals of six basic trig functions}} + C$$

$$= \boxed{-\frac{1}{5}\ln\left|\csc\left(5\theta\right) + \cot\left(5\theta\right)\right| + C}$$

4. $\int 2\tan x \cdot \ln\left(\cos x\right)dx$

$$\int 2\tan x \cdot \ln\left(\cos x\right)dx$$

$$u = \ln\left(\cos x\right)$$

$$du = \frac{1}{\cos x}\left(-\sin x\right)dx$$

$$du = -\tan x\,dx$$

$$-\frac{du}{\tan x} = dx$$

$$= \int 2\tan x \cdot u\left(-\frac{du}{\tan x}\right)$$

$$= -2\int u\,du = -2\left(\tfrac{1}{2}u^2\right) + C$$

$$= \boxed{-\left(\ln\left(\cos x\right)\right)^2 + C}$$

5. $\int \frac{x^3-3x}{x^2-1}dx$

$$\begin{array}{r} x \\ \left(x^2-1\right)\overline{)x^3-3x} \\ \underline{x^3-x} \\ -2x \end{array} \Rightarrow x-\frac{2x}{x^2-1}$$

$$=\int x\,dx-\int\frac{2x}{x^2-1}dx$$

$$=\tfrac{1}{2}x^2-\int\frac{1}{\underbrace{x^2-1}_{u}}\underbrace{\left(2xdx\right)}_{du}$$

$$=\tfrac{1}{2}x^2-\int\frac{1}{u}du=\tfrac{1}{2}x^2-\ln|u|+C$$

$$=\boxed{\tfrac{1}{2}x^2-\ln\left|x^2-1\right|+C}$$

6. $\int\frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

$$u=\sqrt{x}\Rightarrow$$

$$\int\frac{e^{\sqrt{x}}}{\sqrt{x}}dx \qquad du=\frac{1}{2\sqrt{x}}dx$$

$$2\sqrt{x}du=dx$$

$$=\int\frac{e^u}{\sqrt{x}}\left(2\sqrt{x}du\right)=2\int e^u\,du$$

$$=2e^u+C=\boxed{2e^{\sqrt{x}}+C}$$

7. $\int x3^{x^2}dx$

$$u=x^2\Rightarrow du=2xdx$$

$$\int x3^{x^2}dx \qquad \frac{du}{2x}=dx$$

$$=\int x3^u\frac{du}{2x}=\frac{1}{2}\int 3^u\,du$$

$$=\frac{1}{2}\cdot\frac{3^u}{\ln 3}=\boxed{\frac{3^{x^2}}{2\ln 3}+C}$$

8. $\int\frac{dx}{\sqrt{1-4x^2}}$

$$u=2x$$

$$\int\frac{dx}{\sqrt{1-4x^2}} \qquad du=2dx$$

$$\frac{du}{2}=dx$$

$$=\int\frac{dx}{\sqrt{1-\left(2x\right)^2}}=\frac{1}{2}\int\frac{du}{\sqrt{1-u^2}}$$

$$=\frac{1}{2}\arcsin u+C$$

$$=\boxed{\frac{1}{2}\arcsin\left(2x\right)+C}$$

9. $\int \frac{dx}{x\sqrt{9x^2-1}}$

$$\int \frac{dx}{x\sqrt{9x^2-1}} \qquad u = 3x \quad du = 3dx \quad \frac{du}{3} = dx$$

$$= \int \frac{du}{u\sqrt{u^2-1}} = \operatorname{arcsec}|u| + C$$

$$= \boxed{\operatorname{arcsec}|3x| + C}$$

10. $\int \frac{\arctan x}{1+x^2}\,dx$

$$u = \arctan x$$

$$\int \frac{\arctan x}{1+x^2}\,dx \qquad du = \frac{1}{1+x^2}\,dx$$

$$\left(1+x^2\right)du = dx$$

$$= \int \frac{u}{1+x^2}\left(1+x^2\right)du = \int u\,du$$

$$= \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\left(\arctan x\right)^2 + C}$$

11. $\int \frac{x+3}{x^2+9}\,dx$

$$= \int \frac{x}{x^2+9}\,dx + \int \frac{3}{x^2+9}\,dx$$

$$= \boxed{\frac{1}{2}\ln\left|x^2+9\right| + \arctan\frac{x}{3} + C}$$

$$\int \frac{x}{x^2+9}\,dx \qquad u = x^2 \Rightarrow du = 2xdx \quad \frac{du}{2x} = dx$$

$$= \int \frac{x}{u+9}\frac{du}{2x} = \frac{1}{2}\int \frac{1}{u+9}\,du$$

$$= \frac{1}{2}\ln|u+9| + C$$

$$= \frac{1}{2}\ln\left|x^2+9\right| + C$$

$$\int \frac{3}{x^2+9}\,dx \qquad a^2 = 9 \quad a = 3 \qquad u^2 = x^2 \quad u = x \quad du = dx$$

$$= 3\int \frac{1}{u^2+a^2}\,du$$

$$= 3\left(\frac{1}{3}\arctan\frac{u}{3} + C\right)$$

$$= \arctan\frac{x}{3} + C$$

12. $\int \frac{dx}{\sqrt{8+2x-x^2}}$

$$\int \frac{dx}{\sqrt{8+2x-x^2}}$$

$$= \int \frac{dx}{\sqrt{8+1-\left(1-2x+x^2\right)}}$$

$$= \int \frac{dx}{\sqrt{9-(x-1)^2}} \qquad a^2 = 9 \Rightarrow a = 3 \quad u = x-1 \quad du = dx$$

$$= \int \frac{dx}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$$

$$= \boxed{\arcsin\frac{x-1}{3} + C}$$

13. $\int \frac{x-1}{\sqrt{x^2-2x}}\,dx$

$$\int \frac{x-1}{\sqrt{x^2-2x}}\,dx \qquad u = x^2 - 2x$$

$$du = (2x-2)\,dx$$

$$\frac{du}{2(x-1)}$$

$$= \int \frac{\cancel{x-1}}{\sqrt{u}}\,\frac{du}{2\cancel{(x-1)}} = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$

$$= \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\left(2u^{1/2}\right) + C$$

$$= \boxed{\sqrt{x^2-2x} + C}$$

14. $\int \frac{3x}{x^2+5}\,dx$

$$\int \frac{3x}{x^2+5}\,dx \qquad u = x^2 + 5$$

$$du = 2x\,dx$$

$$\frac{du}{2x} = dx$$

$$= \int \frac{3x}{u}\,\frac{du}{2x} = \frac{3}{2}\int \frac{1}{u}\,du =$$

$$= \frac{3}{2}\ln|u| + C = \boxed{\frac{3}{2}\ln\left|x^2+5\right| + C}$$

15. $\int \frac{2x-5}{x^2+2x+2}\,dx$

$$= \int \frac{2x-5+7-7}{x^2+2x+2}\,dx$$

$$= \int \frac{(2x+2)-7}{x^2+2x+2}\,dx$$

$$= \int \frac{(2x+2)}{x^2+2x+2}\,dx - \int \frac{7}{x^2+2x+2}\,dx$$

$$= \boxed{\begin{array}{l}\ln\left|x^2+2x+2\right| \\ \qquad -7\arctan(x+1) + C\end{array}}$$

$$\int \frac{(2x+2)}{x^2+2x+2}\,dx$$

$$= \int \frac{1}{\underbrace{x^2+2x+2}_{u}}\underbrace{(2x+2)\,dx}_{du}$$

$$= \int \frac{1}{u}\,du = \ln|u| + C$$

$$= \ln\left|x^2+2x+2\right| + C$$

$$\int \frac{7}{x^2+2x+2}\,dx$$

$$= \int \frac{7}{\left(x^2+2x+1\right)+1}\,dx$$

$$= \int \frac{7}{(x+1)^2+1}\,dx$$

$$= 7\arctan(x+1) + C$$

Find each definite integral.

16. $\int_1^3 \frac{x+3}{x^2+6x}dx$

$$\int_1^3 \frac{x+3}{x^2+6x}dx$$

$u=x^2+6x$

$du=(2x+6)dx$

$\frac{du}{2(x+3)}=dx$

$$=\int_7^{27} \frac{\cancel{x+3}}{u}\,\frac{du}{2(\cancel{x+3})}$$

$$=\frac{1}{2}\int_7^{27}\frac{1}{u}du=\frac{1}{2}\left[\ln u\right]_7^{27}$$

$$=\boxed{\frac{1}{2}\left[\ln 27-\ln 7\right]}$$

17. $\int_0^\pi \tan\frac{\theta}{3}d\theta$

$$\int_0^\pi \tan\frac{\theta}{3}d\theta$$

$u=\frac{\theta}{3}$

$du=\frac{1}{3}d\theta$

$$=\int_0^{\pi/3}\tan u(3du)=3\int_0^{\pi/3}\tan u\,du$$

$$=3\left[-\ln|\cos u|\right]_0^{\pi/3}$$

$$=3\left[-\ln|\cos(\pi/3)|+\ln|\cos(0)|\right]$$

$$=3\left[-\ln|\cos(\pi/3)|+\ln|1|\right]$$

$$=\boxed{3\left[-\ln|\cos(\pi/3)|\right]}$$

18. $\int_{-3}^{-1}\frac{dx}{x^2+6x+13}$

$$\int_{-3}^{-1}\frac{dx}{x^2+6x+13}$$

$$=\int_{-3}^{-1}\frac{dx}{(x^2+6x+9)+4}$$

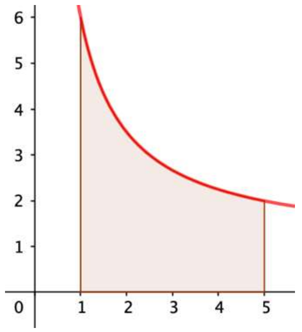
$$=\int_{-3}^{-1}\frac{dx}{(x+3)^2+4}$$

$$=\frac{1}{2}\left[\arctan\left(\frac{x+3}{2}\right)\right]_{-3}^{-1}$$

$$=\frac{1}{2}\left[\arctan\left(\frac{2}{2}\right)-\arctan\left(\frac{0}{2}\right)\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{4}\right]=\boxed{\frac{\pi}{8}}$$

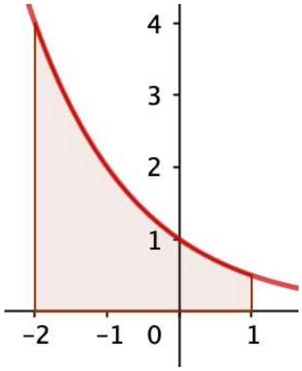
19. Find the area of the region bounded by the graphs of the equations $y=\frac{x+5}{x}$, $x=1$, $x=5$, $y=0$. Express your answer in natural logarithmic form.



$$A=\int_1^5\frac{x+5}{x}dx=\int_1^5 1dx+\int_1^5\frac{5}{x}dx=\left[x\right]_1^5+\left[5\ln x\right]_1^5$$

$$=\left[5-4\right]+\left[5\ln 5-5\ln 1\right]=\boxed{4+5\ln 5}$$

20. Find the area bounded by the function $f(x)=2^{-x}$, the x-axis, $x=-2$, and $x=1$.



$$A=\int_{-2}^{1}2^{-x}\,dx \qquad u=-x\Rightarrow du=-dx$$

$$=\int_{2}^{-1}2^{u}(-du)=\int_{-1}^{2}2^{u}\,du=\left[\frac{2^{u}}{\ln 2}\right]_{-1}^{2}=\frac{1}{\ln 2}\left[2^{2}-2^{-1}\right]$$

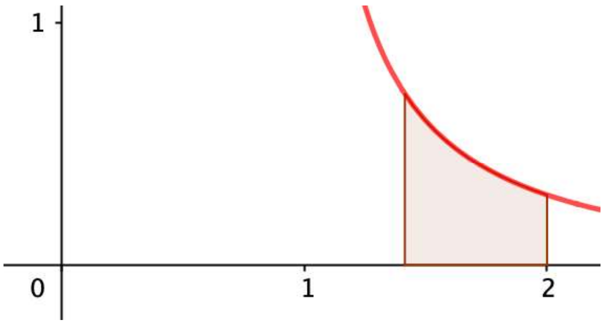
$$=\frac{1}{\ln 2}\left[4-\frac{1}{2}\right]=\boxed{\frac{7}{2\ln 2}}$$

21. Let $f(x)=\dfrac{e+\ln x}{x^{2}}$. Find the average rate of change of f from $x=1$ to $x=e$.

$$AROC=\frac{f(e)-f(1)}{e-1}=\frac{\frac{e+\ln e}{e^{2}}-\frac{e+\ln 1}{1^{2}}}{e-1}=\frac{\frac{e+1}{e^{2}}-\frac{e+0}{1}}{e-1}=\frac{\frac{e+1}{e^{2}}-e}{e-1}=\boxed{\frac{e+1-e^{3}}{(e-1)e^{2}}}$$

22. Find the area of the region bounded by the graphs of the equations.

$$f(x)=\frac{1}{x\sqrt{x^2-1}},\ y=0,\ x=\frac{2}{\sqrt{2}},\ x=2$$

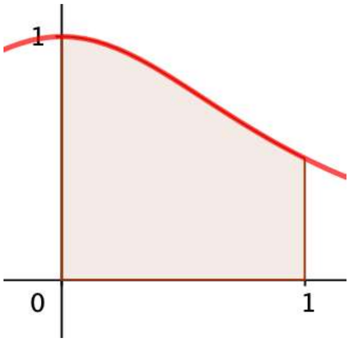


$$A=\int_{\sqrt{2}}^{2}\frac{1}{x\sqrt{x^2-1}}dx$$

$$=\left[\operatorname{arcsec}|x|\right]_{\sqrt{2}}^{2}=\boxed{\operatorname{arcsec}(2)-\operatorname{arcsec}\left(\sqrt{2}\right)}$$

23. (a) Find the area of the region bounded by the graphs of the equations.

$$f(x)=\frac{1}{1+x^2},\ y=0,\ x=0,\ x=1$$



$$A=\int_{0}^{1}\frac{1}{1+x^2}dx=\left[\arctan x\right]_0^1=\arctan 1-\arctan 0=\boxed{\frac{\pi}{4}}$$

You may skip parts (b) and (c).

(b) Find the average value of f on the interval $[0,1]$. **This concept will be taught in Unit 8.**

$$AVG=\frac{1}{1-0}\int_0^1\frac{1}{1+x^2}dx=\int_0^1\frac{1}{1+x^2}dx=\left[\arctan x\right]_0^1=\arctan 1-\arctan 0=\boxed{\frac{\pi}{4}}$$

(c) For what value of x does $f(x)$ take on the average value? **This concept will be taught in Unit 8.**

$$f(c)=\frac{\pi}{4}\Rightarrow\frac{1}{1+c^2}=\frac{\pi}{4}\Rightarrow 1+c^2=\frac{4}{\pi}\Rightarrow c^2=\frac{4}{\pi}-1\Rightarrow\boxed{c=\sqrt{\frac{4}{\pi}-1}=\sqrt{\frac{4-\pi}{\pi}}}$$

You may skip this problem. This concept will be taught in Unit 7.

24. The number of plankton growing in a controlled research vat is changing at the rate of $\frac{dP}{dt}=\frac{1250}{3+0.15t}$ where *t* is measured in weeks. If the initial amount of plankton ($t = 0$) is 2400, write an equation that will give the population of the plankton at time *t* and use it to find the number of plankton (to the nearest whole number) after 2 months ($t = 8$). [The answer will be 5,204 plankton.]

$$\frac{dP}{dt}=\frac{1250}{3+0.15t}\Rightarrow dP=\frac{1250}{3+0.15t}dt$$

$$P=\int\frac{1250}{3+0.15t}dt=1250\left(\frac{1}{0.15}\right)\int\frac{1}{\underbrace{3+0.15t}_{u}}\underbrace{(0.15dt)}_{du}=\frac{125000}{15}\int\frac{1}{u}du$$

$$=\frac{125000}{15}\ln u+C=\frac{125000}{15}\ln(3+0.15t)+C$$

$$P(0)=2400\Rightarrow 2400=\frac{125000}{15}\ln(3)+C\Rightarrow C=2400-\frac{125000}{15}\ln(3)=-6755.1024\ldots$$

$$P(t)=\boxed{\frac{125000}{15}\ln(3+0.15t)-6755.1024\ldots}$$

$$P(8)=\frac{125000}{15}\ln(3+0.15(8))-6755.1024\ldots=5203.9353\ldots\approx\boxed{5204\,\text{plankton}}$$

25. Let $F(x)$ be an antiderivative of $f(x)=\frac{2(\ln x)^2}{e^x}$. If $F(1)=2$, use the Fundamental Theorem of Calculus and your calculator to find $F(5)$. [The answer will be 2.348.]

$$\int_1^5 f(x)\,dx=F(5)-F(1)\Rightarrow F(1)+\int_1^5 f(x)\,dx=F(5)$$

$$F(5)=2+\int_1^5\frac{2(\ln x)^2}{e^x}dx=\boxed{2.3483\ldots}$$