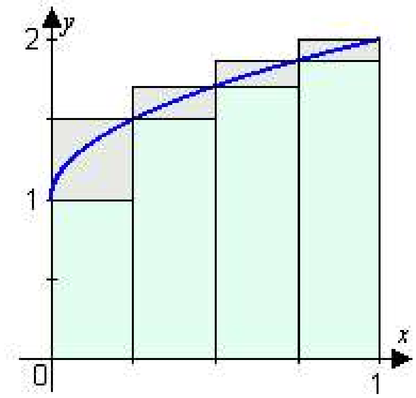
UNIT 6 REVIEW – Integration and Accumulations of Change

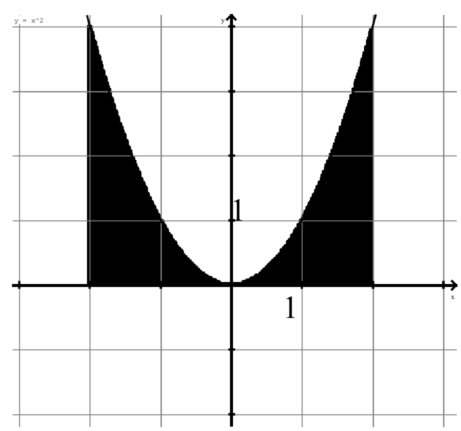
Part I. Multiple Choice: Graphing Calculator Permitted



1. The graph to the right shows upper and lowers sums for the function $f(x)=\sqrt{x}+1$ using 4 sub-intervals. Which of the following correctly approximates the sum using the right hand endpoint?

(A) 1.768 **(B)** 1.518
(C) 1.268 **(D)** 3.030

$$A_R \approx \frac{1}{4}\left[\left(\sqrt{\frac{1}{4}}+1\right)+\left(\sqrt{\frac{1}{2}}+1\right)+\left(\sqrt{\frac{3}{4}}+1\right)+\left(\sqrt{1}+1\right)\right]$$
$$\approx \frac{1}{4}(1.5+1.7071+1.8660+2)$$
$$\approx 1.768$$



2. Which of the following definite integrals represents the area of the shaded region? Each box represents one unit.

(A) $\int_{1}^{-1} x^2 dx$ **(B)** $\int_{-1}^{1} (x^2-1)dx$

(C) $\int_{0}^{2} x^2 dx$ **(D)** $2\int_{0}^{2} x^2 dx$

While $\int_{-2}^{2} x^2\, dx$ is a popular way to depict this area, it is not among the choices. Instead we can integrate from 0 to 2 and double the result.

3. The lower sum of $f(x)=\sqrt{x}$ on the interval $[0,1]$ with four equal subintervals is

(A) 0.768 **(B)** 0.667
(C) 0.518 **(D)** 0.25

$$A_L \approx \frac{1}{4}\left(\sqrt{0}+\sqrt{.25}+\sqrt{.5}+\sqrt{.75}\right)$$
$$\approx 0.518$$

4. Let $f(x)$ be a continuous function. What is the value of $\int_3^{15} f(x)\,dx$ if it is known that $\int_1^5 f(3x)\,dx = 7$?

(A) $\frac{7}{3}$ **(B)** 7

(C) 21 **(D)** $\frac{3}{7}$

$u = 3x \rightarrow du = 3\,dx$

$u(1) = 3; \quad u(5) = 15$

$\int_1^5 f(3x)\,dx = 7 = \frac{1}{3}\int_3^{15} f(u)\,du$

$\int_3^{15} f(u)\,du = 21$

5. Which of the following integral expressions CANNOT be found using any of the basic integration formulas or the techniques that we've discussed in class throughout Unit 6?

(A) $\int \frac{x^5}{1+x^4}\,dx$ **(B)** $\int \frac{x^3}{1+x^4}\,dx$

(C) $\int \frac{x}{1+x^4}\,dx$ **(D)** $\int \frac{1}{1+x^4}\,dx$

(A) can be integrated using long division.

(B) can be integrated using u substitution resulting in a natural log

(C) can be integrated resulting in an arctan form

(D) cannot be integrated using any technique discussed in Unit 6

Part II. Free Response: Graphing Calculator Permitted.

6. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

a. How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)? Give your answer to the nearest whole number.

$$\int_0^5 E(t) \approx 153.458$$

Approximately 153 fish enter the lake from midnight to 5 A.M.

b. What is the total number of fish in the lake at 5 A.M. ($t = 5$)?

$$\int_0^5 \left[E(t) - L(t)\right] dt \approx 123.163$$

c. Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ? Explain your reasoning.

$E'(5) \approx -6.801$ fish per hour per hour

$L'(5) \approx 3.921$ fish per hour per hour

$E'(5) - L'(5) = -10.723 < 0$

Since $E'(5) - L'(5) < 0$, the rate of change in the number of fish in the lake is decreasing at time $t = 5$ hours.

d. At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

Let the number of fish in the lake be represented by $F(t) = \int_0^t \left[E(x) - L(x)\right] dx$.

$F'(t) = E(t) - L(t)$

$F'(t) = E(t) - L(t) = 0$ when $t = 6.204$

t	$F(t)$
0	0
6.204	$\int_0^{6.204} \left[E(t) - L(t)\right] dt \approx 135.015$
8	$\int_0^8 \left[E(t) - L(t)\right] dt \approx 80.920$

Part III. Multiple Choice: No calculator is permitted.

7. Choose the correct statement(s) given that $\int_0^9 f(x)\,dx = 11$ and $\int_3^9 f(x)\,dx = -1$.

I. $\int_9^3 f(x)\,dx = 1$ **II.** $\int_3^0 f(x)\,dx = 6$ **III.** $\int_0^3 f(x)\,dx = 12$

(A) I only **(B)** II only **(C)** III only **(D)** I and II **(E) I and III**

I is correct because when the boundaries of integration are reversed, the answer takes the opposite sign.

Looking at II, we know $\int_0^3 f(x)\,dx = \int_0^9 f(x)\,dx - \int_3^9 f(x)\,dx$ by the additive integral property. This would result in $\int_0^3 f(x)\,dx = 11-(-1) = 12$. $\int_3^0 f(x)\,dx = -12$ and not 6.

8. The limit of the right-hand sum given by $\lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\right]$ represents the area under which function (and above the x-axis) on which interval?

(A) $f(x) = x^3$ on $[1,2]$ **(B)** $f(x) = x^3$ on $[0,2]$

(C) $f(x) = x^3$ on $[0,1]$ **(D)** $f(x) = \frac{1}{x^3}$ on $[0,1]$

width $= \frac{b-a}{n} = \frac{1}{n}$. This eliminates Choice (B).

Because there is no constant added within $\left(\frac{i}{n}\right)^3$, we can eliminate Choice (A).

The function under which we are finding the area must be $f(x) = x^3$, which means the correct answer choice must be (C).

9. If the substitution $u = x - 1$ is made, then the integral $\int_2^5 x\sqrt{x-1}\,dx =$

(A) $\int_2^5 (u+1)\sqrt{u}\,du$ **(B)** $\int_1^4 (u+1)\sqrt{u}\,du$

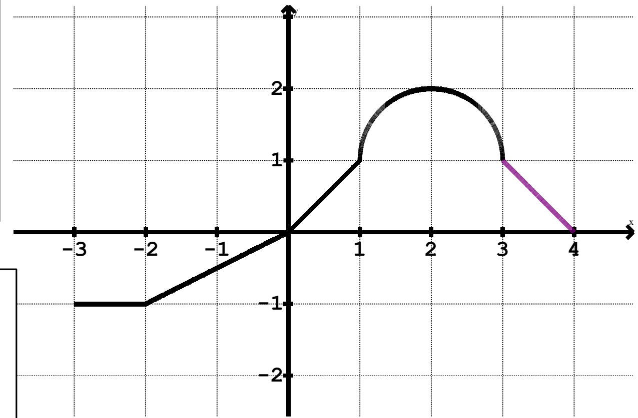
(C) $\int_1^4 u\sqrt{u-1}\,du$ **(D)** $\int_2^5 u\sqrt{u-1}\,du$

$u = x-1 \rightarrow x = u+1$

$du = dx$

$\int_2^5 x\sqrt{x-1}\,dx = \int_1^4 (u+1)\cdot\sqrt{u}$

Use the graph of $f(x)$ below for problems **10** and **11.**



10. Find $\int_1^4 f(x)\,dx$.

(A) $\dfrac{\pi+5}{2}$ **(B)** $\dfrac{5}{2}+\pi$

(C) $\pi+3$ **(D)** $\dfrac{\pi}{2}+3$

$$\int_1^4 f(x)\,dx =$$
$$\frac{1}{2}\pi(1)^2+(2)(1)+\frac{1}{2}(1)(1)$$
$$=\frac{\pi}{2}+2+\frac{1}{2}=\frac{\pi+5}{2}$$

11. Find $\int_{-3}^{1} f(x)\,dx$.

(A) -2.5 **(B)** -1.5

(C) 3 **(D)** 2.5

$$\int_{-3}^{1} f(x)\,dx =$$
$$-(1)(1)-\frac{1}{2}(2)(1)+\frac{1}{2}(1)(1)$$
$$=-1-1+\frac{1}{2}=-\frac{3}{2}$$

12. If the substitution $u=\cos 3x$ is made, then the integral $\int_0^{\frac{\pi}{6}} \cos^3 3x \sin 3x dx =$

(A) $-\dfrac{1}{3}\int_0^{\frac{\pi}{6}} u^3\,du$ **(B)** $\dfrac{1}{3}\int_0^{\pi} u^3\,du$

(C) $-\dfrac{1}{3}\int_1^{0} u^3\,du$ **(D)** $-\dfrac{1}{3}\int_0^{1} u^3\,du$

$$u=\cos 3x \rightarrow du=-3\sin 3x\,dx$$
$$u\left(\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{2}\right)=0$$
$$u(0)=\cos(0)=1$$
$$\int_0^{\frac{\pi}{6}} \cos^3 3x \sin 3x dx = -\frac{1}{3}\int_1^0 u^3\,du$$

13. Consider $F(x)=\int_1^{2x} \dfrac{1}{1+t^3}\,dt$. Find $F'(x)$.

(A) $\dfrac{1}{1+8x^2}$ **(B)** $\dfrac{1}{1+2x^2}$

(C) $\dfrac{2}{1+8x^3}-\dfrac{1}{2}$ **(D)** $\dfrac{2}{1+8x^3}$

$$F'(x)=\frac{1}{1+(2x)^3}\cdot\frac{d}{dx}[2x]$$
$$=\frac{1}{1+8x^3}\cdot 2$$

14. $\int \frac{x^2+1}{x-2}\,dx =$

(A) $x+2+\frac{5}{x-2}+C$

(B) $\frac{\frac{x^2}{2}+2x+5x}{\frac{x^2}{2}-2x}+C$

(C) $\frac{x^2}{2}+2x+5\ln|x-2|+C$

(D) $\frac{\frac{x^3}{3}+x}{\frac{x^2}{2}-2x}+C$

Begin using polynomial long division.

$$\begin{array}{r} x+2 \\ x-2\,\overline{)\,x^2+0x+1} \\ -(x^2-2x) \\ \hline 2x+1 \\ -(2x-4) \\ \hline 5 \end{array}$$

$$\int \frac{x^2+1}{x-2}\,dx = \int\left(x+2+\frac{5}{x-2}\right)dx$$
$$= \frac{x^2}{2}+2x+5\ln|x-2|+C$$

15. $\int \frac{dx}{x^2+8x+20} =$

(A) $\frac{1}{2}\arctan(x-4)+C$

(B) $\frac{1}{2}\arctan\left(\frac{x+4}{2}\right)+C$

(C) $\ln|x^2+8x-20|+C$

(D) $\arcsin\left(\frac{x+4}{2}\right)+C$

$$\int \frac{dx}{x^2+8x+20} = \int \frac{dx}{x^2+8x+16+20-16}$$
$$= \int \frac{dx}{(x+4)^2+4}$$
$$= \frac{1}{2}\arctan\left(\frac{x+4}{2}\right)+C$$

16. Find $y=f(x)$ if $f''(x)=3x+2$, $f'(2)=13$, $f(0)=1$.

(A) $f(x)=\frac{3x^2}{2}+2x+3$

(B) $f(x)=\frac{x^3}{2}+x^2+3x-1$

(C) $f(x)=\frac{x^3}{2}+x^2+3x+1$

(D) $f(x)=\frac{3x^3}{2}+2x^2+x+1$

$f'(x)=\int f''(x)\,dx=\int(3x+2)\,dx=\frac{3x^2}{2}+2x+C_1$

Use $f'(2)=13$ to find C_1

$13=\frac{3(2)^2}{2}+2(2)+C_1 \rightarrow C_1=13-6-4=3$

Therefore, $f'(x)=\frac{3x^2}{2}+2x+3$

$f(x)=\int f'(x)\,dx=\int\left(\frac{3x^2}{2}+2x+3\right)dx=\frac{x^3}{2}+x^2+3x+C_2$

Use $f(0)=1$ to find C_2

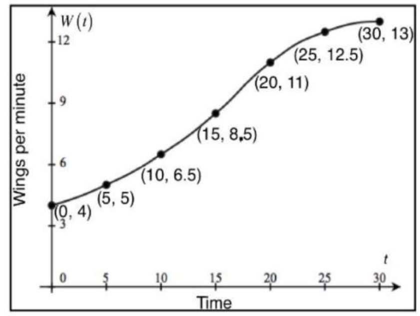
$1=\frac{(0)^3}{2}+(0)^2+3(0)+C_2 \rightarrow C_2=1$

Therefore, $f(x)=\frac{x^3}{2}+x^2+3x+1$

Part IV. Free Response: No calculator is permitted.

Be sure to show all your work. Point values are given in ( ).

17. The Wing Bowl is an annual eating contest founded by the Morning Show on Philadelphia's WIP Radio as a celebration of gluttony. The event, which attracts thousands of spectators, is a contest to see who can eat the most chicken wings in 30 minutes. The rate of wing consumption, in wings per minute, recorded during a past Wing Bowl of the champion El Wingador, is given by a twice-differentiable and strictly increasing function W of time t. The graph of W for the time interval $0 \le t \le 30$ minutes is shown.



a. Approximate the value of $\int_0^{30} W(t)\,dt$ using a **left** Riemann sum with six subintervals indicated by the data from the graph.

$$\int_0^{30} W(t)dt \approx \frac{30-0}{6}\left(W(0)+W(5)+W(10)+W(15)+W(20)+W(25)\right)$$
$$\approx \frac{30-0}{6}\left(4+5+6.5+8.5+11+12.5\right)$$
$$\approx 5(47.5)$$
$$\approx 237.5$$

b. Is the numerical approximation found in part (a) above less than or greater than the actual value of $\int_0^{30} W(t)\,dt$? Justify your answer.

The approximation is less than the actual value of $\int_0^{30} W(t)\,dt$ because $W(t)$ is increasing.

c. Approximate the value of $\int_0^{30} W(t)\,dt$ using a **<u>trapezoidal</u>** sum with six subintervals indicated by the data from the graph.

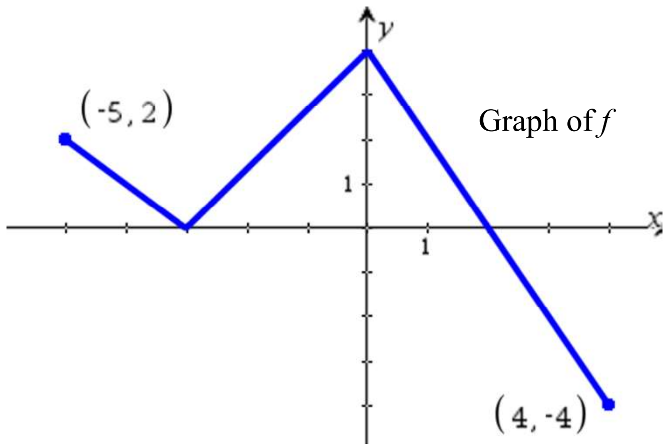
$$\int_0^{30} W(t) \approx \frac{30-0}{2(6)}\left(W(0)+2\cdot W(5)+2\cdot W(10)+2\cdot W(15)+2\cdot W(20)+2\cdot W(25)+W(30)\right)$$
$$\approx \frac{30}{12}(4+2\cdot 5+2\cdot 6.5+2\cdot 8.5+2\cdot 11+2\cdot 12.5+13)$$
$$\approx \frac{5}{2}(4+10+13+17+22+25+13)$$
$$\approx \frac{5}{2}(104)$$
$$\approx 260$$

d. Explain the meaning of the value of $\int_0^{30} W(t)\,dt$ using correct units. **(1 point)**

$\int_0^{30} W(t)\,dt$ signifies the total number of chicken wings El Wingador consumed in the 30-minute time period.

Unit 6 – Review Problems (The Integral as an Accumulator)

The following problem appeared on the 2014 AP Calculus AB Exam.



1.) The function f is defined on the closed interval $[-5,4]$. The graph of f consists of three line segments and is shown in the figure to the above. Let g be the function defined by $g(x)=\int_{-3}^{x} f(t)\,dt$.

a. Find $g(3)$.

$$g(3)=\int_{-3}^{3} f(t)\,dt=\frac{1}{2}(5)(4)-\frac{1}{2}(1)(2)=10-1=9$$

b. On what open intervals contained in $-5<x<4$ is the graph of g both increasing and concave up?

$g'(x)=f(x)>0$ on $[-5,2]$

$g''(x)=f'(x)>0$ on $[-3,0]$

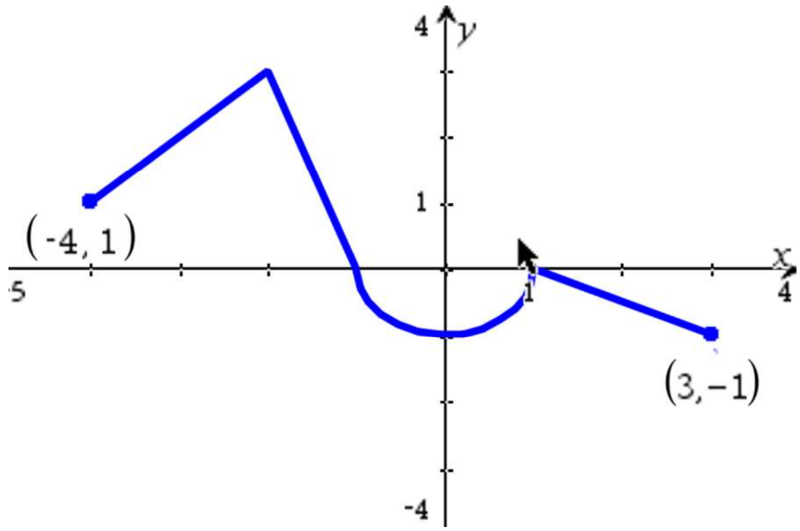
The graph of g is both increasing and concave up on the interval $[-3,0]$.

c. The function $h(x)$ is defined by $h(x)=\dfrac{g(x)}{5x}$. Find $h'(3)$.

$$h'(x)=\frac{g'(x)\cdot 5x-g(x)\cdot 5}{(5x)^2}$$

$$h'(3)=\frac{g'(3)\cdot 5(3)-g(3)\cdot 5}{(5\cdot 3)^2}=\frac{(-2)(15)-(9)(5)}{225}=-\frac{75}{225}=-\frac{1}{3}$$

The following problem appeared on the 2012 AP Calculus AB Exam.



2.) Let f be the continuous function defined on $[-4,3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x)=\int_1^x f(t)\,dt$.

a. Find the values of $g(2)$ and $g(-2)$.

$g(2)=\int_1^2 f(t)\,dt=-\frac{1}{2}(1)\left(-\frac{1}{2}\right)=\frac{1}{4}$

$g(-2)=\int_1^{-2} f(t)\,dt=-\int_{-2}^{1} f(t)\,dt=-\left(\frac{1}{2}(1)(3)-\frac{1}{2}\pi(1)^2\right)=-\left(\frac{3}{2}-\frac{\pi}{2}\right)=\frac{\pi-3}{2}$

b. For each of $g'(-3)$ and $g''(-3)$, find the value or state that it doesn't exist.

$g'(x)=f(x)$ $\quad$ $g''(x)=f'(x)$

$g'(-3)=f(-3)=2$ $\quad$ $g''(-3)=f'(-3)=1$

c. Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

g has a horizontal tangent when $g'(x)$ or $f(x)=0$ which occurs at $x=-1$ and $x=1$.

g has a relative maximum at $x=-1$ because $g'(x)=f(x)$ changes signs from positive to negative at $x=-1$.

g has neither a relative maximum nor minimum at $x=1$ because $g'(x)=f(x)$ does not change signs at $x=1$.

b. For $-4<x<3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

$g''(x)=f'(x)=0$ at $x=0$ $\quad$ $g''(x)=f'(x)$ is undefined at $x=-2$ and $x=1$.

g has a point of inflection at $x=-2,\ x=0,$ and $x=1$ because $g''(x)=f'(x)$ changes signs at those values.