Skill Builder: Unit 7 – Differential Equations (Big 10)

Complete the ten problems below. Once you complete each problem, cross off the appropriate value in the box below according to the instructions. If done correctly, you will have all 10 numbers crossed off with no repeats.

0	1	2	3	4	5	6	7	8	9

A) Consider the differential equation $\frac{dy}{dx} = \frac{3y^2}{\cos(\pi x)}$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(3) = -1$. The equation of the line tangent to graph of f at $(3,-1)$ can be written as $y = mx + b$. Find b.

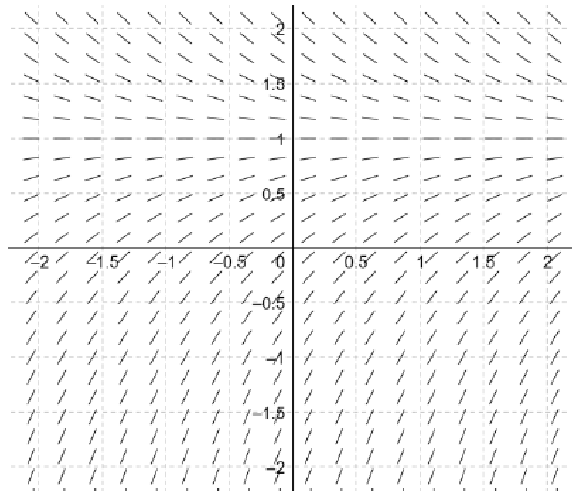
$$\left.\frac{dy}{dx}\right|_{(3,-1)} = \left.\frac{3y^2}{\cos(\pi x)}\right|_{(3,-1)} = \frac{3}{\cos(3\pi)} = \frac{3}{-1} = -3$$

$$y - (-1) = -3(x-3)$$

$$y = -3x + 8$$

B) A slope field for a given differential equation is shown to the right. There is a horizontal line with equation $y = c$ that satisfies the differential equation. Find the value of c.

The only place on the slope field graph that features such a horizontal asymptote is at $y = 1$.



C) Consider the differential equation $\frac{dy}{dx} = \frac{2y}{x^2}$. Let $y = h(x)$ be the particular solution to the differential equation through $\left(2, \frac{5}{e}\right)$. Find $\lim_{x\to\infty} h(x)$.

$$\frac{1}{y}dy = \frac{2}{x^2}dx$$

$$\int \frac{1}{y}dy = \int \frac{2}{x^2}dx$$

$$\ln|y| = -\frac{2}{x} + C$$

Use $\left(2, \frac{5}{e}\right)$ to find C

$$\ln\left|\frac{5}{e}\right| = -\frac{2}{2} + C$$

$$C = 1 + \ln\left(\frac{5}{e}\right)$$

$$\ln|y| = -\frac{2}{x} + 1 + \ln\left(\frac{5}{e}\right)$$

$$e^{\ln|y|} = e^{-\frac{2}{x}+1+\ln\left(\frac{5}{e}\right)}$$

$$h(x) = y = \frac{5}{e}e^{-\frac{2}{x}+1}$$

$$\lim_{x\to\infty} \frac{5}{e}e^{-\frac{2}{x}+1}$$

$$= \frac{5}{e}\cdot e^0 = \frac{5}{e}$$

D) The differential equation $\frac{dy}{dx} = \frac{(3-x)^2}{y}$ has the particular solution $y = f(x)$ with initial condition $f(-1) = 4$. Find the slope of the tangent line to the graph of f at $x = -1$.

$$\left.\frac{dy}{dx}\right|_{(-1,4)} = \frac{(3-(-1))^2}{4} = \frac{16}{4} = 4$$

E) Let $y = g(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = \frac{3\sin(\pi x)}{2y}$ with initial condition $g\left(\frac{1}{2}\right) = 3$. Then $g(x) = \sqrt{a + b\cos(\pi x)}$ where a and b are constants. Find a.

$$y\,dy = \frac{3}{2}\sin(\pi x)dx$$

$$\int y\,dy = \int \frac{3}{2}\sin(\pi x)dx$$

$$\frac{y^2}{2} = -\frac{3}{2\pi}\cos(\pi x) + C$$

Use $\left(\frac{1}{2}, 3\right)$ to find C

$$\frac{3^2}{2} = -\frac{3}{2\pi}\cos\left(\frac{\pi}{2}\right) + C$$

$$C = \frac{9}{2}$$

$$\frac{y^2}{2} = -\frac{3}{2\pi}\cos(\pi x) + C$$

$$\frac{y^2}{2} = -\frac{3}{2\pi}\cos(\pi x) + \frac{9}{2}$$

$$y^2 = -\frac{3}{\pi}\cos(\pi x) + 9$$

$$g(x) = y = \sqrt{9 - \frac{3}{\pi}\cos(\pi x)}$$

$$a = 9$$

F) The non separable differential equation $\frac{dy}{dx} = 8x - 2y$ has a **linear** particular solution of the form $y = mx + b$. Find $m + b$.

Because the solution is linear, $\frac{d^2y}{dx^2} = 0.$

For our given $\frac{dy}{dx}$, $\frac{d^2y}{dx^2} = 8 - 2\frac{dy}{dx} = 8 - 2(8x - 2y)$

So, $8 - 2(8x - 2y) = 0$

$$8 - 16x + 4y = 0$$

$$4y = 16x - 8$$

$$y = 4x - 2$$

$$m + b = 4 + (-2) = 2$$

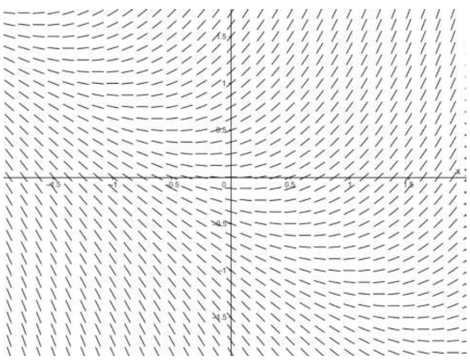
G) Which of the following could be a slope field for the differential equation $\frac{dy}{dx} = x - y$?

Upper right corner - $(2,2)$: $\left.\frac{dy}{dx}\right|_{(2,2)} = 2 - 2 = 0$ This eliminates 3 and 9.

Lower right corner - $(2,-2)$: $\left.\frac{dy}{dx}\right|_{(2,-2)} = 2 - (-2) = 4$ This further eliminates 5.

7 depicts the correct slope field

3.



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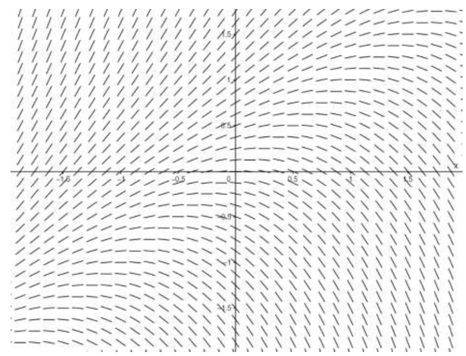

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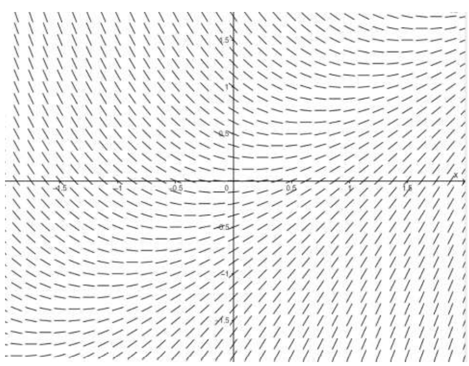

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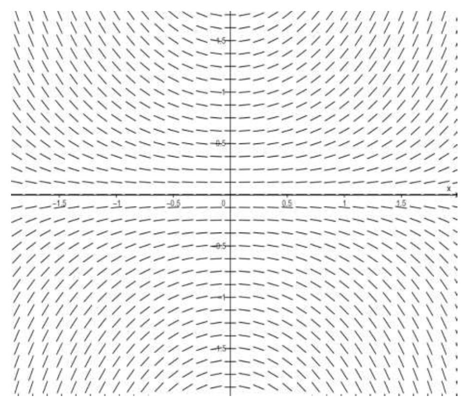

H) Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = \frac{1}{3}(9 - y)$ with the initial condition $f(1) = 3$. Use the tangent line to the graph of f at $(1,3)$ to approximate $f(2.5)$.

$$\left.\frac{dy}{dx}\right|_{(1,3)} = \frac{1}{3}(9-3) = 2 \quad \rightarrow \quad y - 3 = 2(x-1) \quad \rightarrow \quad f(2.5) \approx y(2.5) = 3 + 2(2.5 - 1)$$

$$= 3 + 2\left(\frac{5}{2} - 1\right) = 3 + 5 - 2 = 6$$

I) For $0 \le t \le 3$ days, the number of weeds in a large garden is given by the function W that satisfies the differential equation $\frac{dW}{dt} = \frac{1}{12}\left(-318 + 24W\right)$. At time $t = 2$ days, there are 20 weeds in the garden.

Find $\frac{d^2W}{dt^2}$ when $W = 14$.

$$\frac{d^2W}{dt^2} = \frac{1}{12}\left(24\frac{dW}{dt}\right) = \frac{1}{12}\left(24\left(\frac{1}{12}(-318 + 24W)\right)\right)$$

$$\left.\frac{d^2W}{dt^2}\right|_{W=14} = \frac{1}{12}\left(24\left(\frac{1}{12}(-318 + 24(14))\right)\right) = 3$$

J) Consider the differential equation $\frac{dy}{dx} = \left(1 - \frac{4}{x^3}\right)(y-1)^2$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = -1$. Find $f(1)$.

$$\frac{1}{(y-1)^2}dy = \left(1 - \frac{4}{x^3}\right)dx$$

$$\int \frac{1}{(y-1)^2}dy = \int\left(1 - \frac{4}{x^3}\right)dx$$

$$\frac{-1}{y-1} = x + \frac{2}{x^2} + C$$

Use $(2,-1)$ to find C

$$\frac{-1}{(-1)-1} = 2 + \frac{2}{2^2} + C$$

$$\frac{1}{2} = 2 + \frac{1}{2} + C$$

$$C = -2$$

$$\frac{-1}{y-1} = x + \frac{2}{x^2} - 2$$

Let $x = 1$ and solve for y.

$$\frac{-1}{y-1} = 1 + \frac{2}{1^2} - 2$$

$$\frac{-1}{y-1} = 1$$

$$(y-1) = -1$$

$$y = 0$$

Special thanks to Bryan Passwater, Speedway HS, Speedway, IN for authoring this activity.