AP Calculus AB

SOLUTIONS

UNIT 7 REVIEW

No Calculators should be used for Problems #1-7.

For Problems 1-7, solve for y if it is reasonable to do so.

1. Find the general solution to the differential equation $2x(y+1)-yy'=0$.

$$2x(y+1)=y\frac{dy}{dx}$$
$$2x\,dx=\frac{y}{y+1}dy$$
$$\int 2x\,dx=\int\frac{y}{y+1}dy$$
$$\int 2x\,dx=\int\left(1-\frac{1}{y+1}\right)dy$$
$$x^2+C=y-\ln|y+1|$$

Long division of $\frac{y}{y+1}$

$$\begin{array}{r} 1-\frac{1}{y+1} \\ y+1\overline{\big)\;\;\;\;\;\;\;\;\;\;y} \\ \underline{-(y+1)} \\ -1 \end{array}$$

2. Find the general solution to the differential equation $xy'+2y=0$.

$$x\frac{dy}{dx}=-2y$$
$$\frac{1}{y}dy=-\frac{2}{x}dx$$
$$\int\frac{1}{y}dy=-\int\frac{2}{x}dx$$
$$\ln|y|=-2\ln|x|+C_0$$

Solve for y

$$e^{\ln|y|}=e^{-2\ln|x|+C_0}$$
$$e^{\ln|y|}=e^{\ln(|x|)^{-2}}\cdot e^{C_0}$$
$$y=\frac{C}{x^2}$$

3. Find the general solution to the differential equation $y\,y'-2e^x=0$.

$$ydy=2e^x\,dx$$
$$\int y\,dy=\int 2e^x\,dx$$
$$\frac{y^2}{2}=2e^x+C_0 \quad \text{or} \quad y^2=4e^x+C$$

4. Find the particular solution to the differential equation $xyy'-\ln x=0$ given the initial condition $y(1)=0$.

$$xy\frac{dy}{dx}=\ln x$$
$$ydy=\frac{\ln x}{x}dx$$
$$\int ydy=\int\frac{\ln x}{x}dx$$
$$\frac{y^2}{2}=\frac{(\ln x)^2}{2}+C$$

Use (1,0) to solve for C

$$\frac{0^2}{2}=\frac{(\ln 1)^2}{2}+C$$
$$C=0$$
$$\frac{y^2}{2}=\frac{(\ln x)^2}{2} \quad \text{or} \quad y^2=(\ln x)^2$$

5. Find the particular solution to the differential equation $y\sqrt{1-x^2}\,y' - x\sqrt{1-y^2} = 0$ given the initial condition $y(0) = 1$.

$$y\sqrt{1-x^2}\,\frac{dy}{dx} = x\sqrt{1-y^2}$$

$$\frac{y}{\sqrt{1-y^2}}\,dy = \frac{x}{\sqrt{1-x^2}}\,dx$$

$$\int \frac{y}{\sqrt{1-y^2}}\,dy = \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$-\frac{1}{2}\cdot 2\cdot\sqrt{1-y^2} = -\frac{1}{2}\cdot 2\cdot\sqrt{1-x^2} + C$$

Use (0,1) to solve for C

$$\sqrt{1-1^2} = \sqrt{1-0^2} + C$$

$$C = -1$$

$$\sqrt{1-y^2} = \sqrt{1-x^2} - 1$$

$$1-y^2 = 1-x^2 - 2\sqrt{1-x^2} + 1$$

$$y^2 = x^2 + 2\sqrt{1-x^2} - 1$$

6. Find the particular solution to the differential equation $(1+x^2)(1+y^2) = xyy'$ given the initial condition $y(1) = 0$.

$$(1+x^2)(1+y^2) = xy\frac{dy}{dx}$$

$$\frac{1+x^2}{x}\,dx = \frac{y}{1+y^2}\,dy$$

$$\int \frac{1+x^2}{x}\,dx = \int \frac{y}{1+y^2}\,dy$$

$$\int \left(\frac{1}{x} + x\right)dx = \int \frac{y}{1+y^2}\,dy$$

$$\ln|x| + \frac{x^2}{2} + C = \frac{1}{2}\ln\left|1+y^2\right|$$

Use (1,0) to solve for C

$$\ln|1| + \frac{1}{2} + C = \frac{1}{2}\ln\left|1+0^2\right|$$

$$C = -\frac{1}{2}$$

So, $\ln|x| + \frac{x^2}{2} - \frac{1}{2} = \frac{1}{2}\ln\left|1+y^2\right|$

$$2\ln|x| + x^2 - 1 = \ln\left|1+y^2\right|$$

Solve for y^2

$$e^{\ln x^2 + x^2 - 1} = e^{\ln\left|1+y^2\right|}$$

$$e^{\ln x^2}\cdot e^{x^2}\cdot e^{-1} = \left|1+y^2\right|$$

$$\left|1+y^2\right| = e^{x^2-1}\cdot x^2$$

$$y^2 = e^{x^2-1}\cdot x^2 - 1$$

7. Find the general solution to the differential equation $e^{2y}y' = x^3$.

$$e^{2y}\frac{dy}{dx} = x^3$$

$$e^{2y}dy = x^3\,dx$$

$$\int e^{2y}dy = \int x^3\,dx$$

$$\frac{1}{2}e^{2y} = \frac{x^4}{4} + C$$

Solve for y

$$e^{2y} = \frac{x^4}{2} + C$$

$$\ln e^{2y} = \ln\left(\frac{x^4}{2} + C\right)$$

$$2y = \ln\left(\frac{x^4}{2} + C\right)$$

$$y = \frac{1}{2}\ln\left(\frac{x^4}{2} + C\right)$$

Free Response – A Calculator is Required.

8. The rate at which people consume Pringles potato chips in the U.S. is given by $P(t) = Ce^{kt}$, where P is measured in millions of cans per year and t *is* measured in years from the beginning of 1990.
The consumption rate triples every 4 years and the consumption rate at the beginning of 1990 was 8 million cans per year. Find C and k.

$P = Ce^{kt}$

Because there were 8,000,000 cans consumed in 1990, we'll let $t = 0$ correspond to 1990, which means that $C = 8,000,000$

Use the fact that P triples when $t = 3$.

$24,000,000 = 8,000,000e^{k4}$

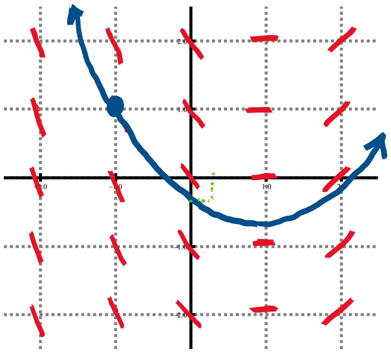
$e^{4k} = 3 \rightarrow \ln e^{4k} = \ln 3 \rightarrow 4k = \ln 3 \rightarrow k = \frac{1}{4}\ln 3$

Free Response – No Calculator May Be Used.

9. Consider the differential equation $\frac{dy}{dx} = x - 1$.

a. Create a slope field for this differentiable equation using 25 points.

b. Sketch a solution to the differential equation on the slope field above through the point $(-1,1)$.



c. Write the equation of a tangent line drawn to $f(x)$ that passes through $(-1,1)$.

$\left.\frac{dy}{dx}\right|_{x=-1} = -1 - 1 = -2$

$y - 1 = -2(x + 1)$

d. Use the tangent line from part (c) to estimate $f(0.3)$

$f(0.3) \approx 1 - 2(0.3 + 1) = 1 - 2.6 = -1.6$

e. Find the particular solution to the differential equation for the initial point $(-1,1)$.

$dy = (x - 1)\,dx$

$\int dy = \int (x - 1)\,dx$

$y = \frac{x^2}{2} - x + C$

Use $(-1,1)$ to find C

$1 = \frac{(-1)^2}{2} - (-1) + C$

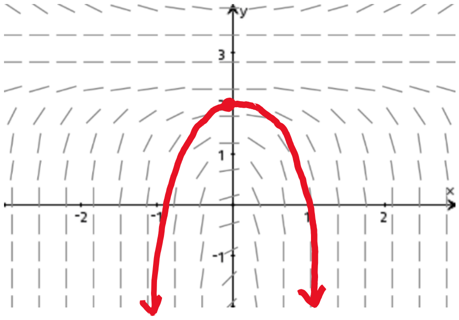
$C = -\frac{1}{2}$

$\therefore\ y = \frac{x^2}{2} - x - \frac{1}{2}$

Free Response – A Calculator is Required.

10. Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}x(y-4)^3$.

a. A slope field for the given differential equation is shown below. Sketch the solution curve that passes through the point $(0,2)$.



b. Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = 0$.

$$\frac{1}{(y-4)^3}dy = \frac{1}{2}x\,dx$$

$$\int \frac{1}{(y-4)^3}dy = \int \frac{1}{2}x\,dx$$

$$\frac{(y-4)^{-2}}{-2} = \frac{x^2}{4} + C$$

$$-\frac{1}{2(y-4)^2} = \frac{x^2}{4} + C$$

$$\frac{1}{(y-4)^2} = -\frac{x^2}{2} + C$$

Use $(1,0)$ to find C

$$\frac{1}{(0-4)^2} = -\frac{1^2}{2} + C$$

$$C = \frac{1}{16} + \frac{1}{2} = \frac{9}{16}$$

$$\therefore\ \frac{1}{(y-4)^2} = -\frac{x^2}{2} + \frac{9}{16}$$

$$\frac{1}{(y-4)^2} = \frac{-8x^2+9}{16}$$

$$(y-4)^2 = \frac{16}{-8x^2+9}$$

$$y-4 = -\sqrt{\frac{16}{-8x^2+9}}$$

$$y = 4 - \sqrt{\frac{16}{-8x^2+9}}$$

Note: When we take the square root of both sides of the equation, we must place a negative sign before the right side in order to obtain a solution that satisfies our initial condition.

Multiple Choice - No Calculator May Be Used

15. If $\frac{dy}{dx} = x + 2xy$ and $y(0) = 1$, find y as a function of x.

a. $y = 3e^{x^2} - 2$ **b.** $y = 3x^2 + 1$ **c.** $y = \frac{3}{2}e^{x^2} - \frac{1}{2}$ **d.** $y = 3\ln(x^2+1) + 1$ **e.** $y = \frac{3}{2}e^{\frac{x^2}{2}} - \frac{1}{2}$

$$\frac{dy}{dx} = x(1+2y)$$

$$\frac{1}{1+2y}dy = x\,dx$$

$$\int \frac{1}{1+2y}dy = \int x\,dx$$

$$\frac{1}{2}\ln|1+2y| = \frac{x^2}{2} + C$$

Use $(0,1)$ to find C

$$\frac{1}{2}\ln|1+2(1)| = \frac{0^2}{2} + C$$

$$C = \frac{1}{2}\ln 3$$

$$\frac{1}{2}\ln|1+2y| = \frac{x^2}{2} + \frac{1}{2}\ln 3$$

$$\ln|1+2y| = x^2 + \ln 3$$

$$e^{\ln|1+2y|} = e^{x^2+\ln 3}$$

$$1+2y = 3e^{x^2} \quad \to \quad y = \frac{3}{2}e^{x^2} - \frac{1}{2}$$

16. The general solution of the differential equation $y dy = -\ x dx$ is a family of

a. circles

b. hyperbolas

c. parallel lines

d. parabolas

e. lines passing through the origin

$$\int y\,dy = -\int x\,dx$$
$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$
$$y^2 = -x^2 + C$$
$$x^2 + y^2 + C$$

17. The differential equation $1\frac{dT}{dt} = 14 - 0.01T$ models the temperature T, above room temperature, of an electric burner. T is measured in degrees Fahrenheit (°F) and $t \geq 0$ is in seconds.

a. Find the general solution of the differential equation. Then solve for T.

$$\frac{1}{14-0.01T}dT = dt$$
$$\int \frac{1}{14-0.01T}dT = \int dt$$
$$-100 \cdot \ln\left|14-0.01T\right| = t + C_0$$
$$\ln\left|14-0.01T\right| = -\frac{t}{100} + C_0$$
$$e^{\ln|14-0.01T|} = e^{-\frac{t}{100}+C_0}$$
$$14-0.01T = e^{-\frac{t}{100}} \cdot e^{C_0}$$
$$14-0.01T = Ce^{-\frac{t}{100}}$$
$$-0.01T = -14 + Ce^{-\frac{t}{100}}$$
$$0.01T = 14 + Ce^{-\frac{t}{100}}$$
$$T = 1400 + Ce^{-\frac{t}{100}}$$

b. When the burner is turned on $(t = 0)$, $T = 70°$. Use this fact to find the value of the constant.

$$T(0) = 70 = 1400 + Ce^{-(0)}$$
$$C = -1330$$

c. The safety standards Board has determined that this type of burner, left on continuously, is unsafe if its temperature can exceed 1500°F. Is this burner safe or unsafe? Justify your answer.

$$\lim_{t\to\infty}\left(1400 - 1330e^{-\frac{t}{100}}\right) = 1400$$

The burner is safe.